EXISTENCE OF UNIVERSAL CONNECTIONS II.*

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1. Introduction. In an earlier paper [3] we proved the existence of universal connections for connections in bundles with a compact Lie group as structure group. In this paper we extend this result to the case of an arbitrary connected Lie group (Theorem 1). The proof of this theorem does not depend on [3]. However, this result does not include Theorem 1 of [3] which is more precise in that it asserts that the canonical connections in the Stiefel bundles themselves are universal for connections in unitary or orthogonal bundles. The latter is useful in some applications.

Since any two connections on a principal bundle differ by a 1-form of the adjoint type, one can reduce the problem of finding a universal connection to one of finding a universal 1-form of the adjoint type (§3). Regarding the latter problem we prove the following more general result (Theorem 2): if \( \rho \) is a finite dimensional representation of a connected Lie group \( G \) and \( n \) and \( p \) are non-negative integers, then there exists a \( n \)-universal \( p \)-form of type \( \rho \). (For the definition of forms of type \( \rho \) see §2.) This problem is essentially one for compact Lie groups (§6) since the structure group of a \( G \)-bundle \( P \) can be reduced to a maximal compact subgroup \( K \) of \( G \), and forms of type \( \rho \) on \( P \) are precisely forms on \( P \) obtained by extending forms of type \( \rho | K \) (restriction of \( \rho \) to \( K \)) on the reduced bundle (§2). It should be remarked, however, that the existence of universal connections for a connected Lie group does not follow immediately from that for compact Lie groups, since not every connection on a \( G \)-bundle is the extension of a connection on a reduced \( K \)-bundle. (For instance, the holonomy group of a connection got by extension will have to be contained in \( K \).) In the case of connections, Theorem 2 seems to be necessary for passing from the compact to the general case. In our procedure, however, Theorem 2 implies at once Theorem 1 without passing through the compact case.

In the case when \( G \) is the orthogonal group \( O(k) \) and \( \rho \) is the natural representation, an \( n \)-universal \( p \)-form is constructed explicitly (§4). If \( G \) is compact we may suppose that \( \rho \) is a representation of \( G \) by orthogonal matrices. This enables one to reduce the compact to the orthogonal case (§5).

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For the notions relating to connections in principal bundles we refer
to [1], [2], [5].

2. Preliminaries. In this section, we first fix our notation and termin-
ology, and then give a canonical way of extending \( p \)-forms of a certain type
to bundles obtained by extension of structure group.

By ‘differentiable’ we always mean ‘indefinitely differentiable.’ All
manifolds, groups, bundles, maps and forms are assumed differentiable. We
assume all our manifolds are paracompact. By a \( p \)-form on a manifold we
mean a covariant tensor of degree \( p \). If \( f: M \to M' \) is a map and \( \alpha \) a \( p \)-form
on \( M' \), \( f^*(\alpha) \) will denote the inverse image of \( \alpha \) by \( f \).

By a \( G \)-bundle we always mean a principal bundle with structure group
\( G \). If \( f: H \to G \) is a homomorphism of groups, \( P_1 \) a \( H \)-bundle, and \( P_2 \) a
\( G \)-bundle, a map \( h: P_1 \to P_2 \) will be called a \( f \)-morphism if \( h(\xi s) = h(\xi)f(s) \)
for every \( \xi \in P_1, s \in H \). When \( H = G \) and \( f \) is the identity map, a \( f \)-morphism
will be called a \( G \)-morphism. If \( \rho \) is a representation of \( G \) in a finite
dimensional vector space \( V \), a \( p \)-form on a \( G \)-bundle with values in \( V \) is said to be
of type \( \rho \) if i) it is equivariant for the action of \( G \) and ii) it annihilates any
\( p \)-tuple of tangent vectors one of which is vertical [2]. If \( h: P_1 \to P_2 \) is a
\( f \)-morphism and \( \alpha \) a \( p \)-form of type \( \rho \) on \( P_2 \), then \( h^*\alpha \) is clearly a \( p \)-form on
\( P_1 \) of type \( \rho \circ f \).

Let \( G_1, G_2 \) be two Lie groups and \( f: G_1 \to G_2 \) a homomorphism. Let \( T_f \)
be the functor which associates to every \( G_1 \)-bundle \( P \) the \( G_2 \)-bundle \( T_f(P) \)
over the same base obtained by extension of the structure group by \( f \). We
recall that the total space of \( T_f(P) \) is the orbit space of \( P \times G_2 \) under
the action of \( G_1 \) given by \( (\xi, g_2)_{(\xi'g_2)g_2} = (\xi g_2, f(g_1)^{-1}g_2) \) for
\( \xi \in P, g_1 \in G_1, g_2 \in G_2 \). The action of \( G_2 \) on \( P \times G_2 \) defined by
\( (\xi, g_2)_{(\xi'g_2)g_2} = (\xi, g_2 g_2') \) for \( \xi \in P, g_2, g_2' \in G_2 \) commutes with the above action of \( G_1 \) and hence \( G_2 \) operates on
\( T_f(P) \) and makes of it a \( G_2 \)-bundle. Moreover, if \( \Phi: P \to P' \) is a \( G_1 \)-morphism,
the \( G_2 \)-morphism \( T_f(\Phi): T_f(P) \to T_f(P') \) is induced by the map \( (\xi, g_2) \to
(\Phi(\xi), g_2) \) of \( P \times G_2 \) into \( P' \times G_2 \).

Let \( q \) be the projection \( P \times G_2 \to T_f(P) \) and \( i_f \) the map \( \xi \to q(\xi, e) \) of \( P \)
into \( T_f(P) \). We then have \( i_f(\xi s) = i_f(\xi)f(s), \xi \in P, s \in G_1 \); i.e., \( i_f \) is a
\( f \)-morphism.

Let \( \rho \) be a finite dimensional representation of \( G_2 \). If \( \beta \) is a \( p \)-form
of type \( \rho \) on \( T_f(P) \), \( i_f^*\beta \) is a \( p \)-form on \( P \) of type \( \rho \circ f \). Conversely, to every
\( p \)-form \( \alpha \) on \( P \) of type \( \rho \circ f \) we can associate in a natural way a \( p \)-form \( T_f(\alpha) \)
of type \( \rho \) on \( T_f(P) \) with \( i_f^*T_f(\alpha) = \alpha \) in the following way. It is easy to
check that the form \( \alpha' \) on \( P \times G_2 \) defined by \( \alpha'_{(\xi, g_2)} = \rho(g_2)^{-1}(p_i^*\alpha)_{(\xi, g_2)} \)
(\rho_1 being the projection \( P \times G_2 \to P \)) is invariant under the action of \( G_1 \) and annihilates any \( p \)-tuple of tangent vectors one of which is vertical. Hence there exists a \( p \)-form \( T_\gamma(\alpha) \) of type \( \rho \) on \( T_\gamma(P) \) such that \( q^*T_\gamma(\alpha) = \alpha' \) and we have \( t^*_\gamma T_\gamma(\alpha) = \alpha \). Moreover, if \( \beta \) is a \( p \)-form of type \( \rho \) on \( T_\gamma(P) \), we have \( T_\gamma(t^*_\gamma \beta) = \beta \).

Finally, the correspondence \( \alpha \to T_\gamma(\alpha) \) is 'functorial' in the following sense: if \( \Phi: P \to P' \) is a \( G_1 \)-morphism and if \( \alpha' \) is a \( p \)-form on \( P' \) of type \( \rho \circ f \), then we have

\[ T_\gamma(\Phi^* \alpha') = (T_\gamma \Phi)^* (T_\gamma \alpha'). \]


**Theorem 1.** Let \( G \) be a connected Lie group and \( n \) a positive integer. Then there exist a principal \( G \)-bundle \( B \) and a connection form \( \gamma_0 \) on \( B \) such that any connection form on a principal \( G \)-bundle \( P \) with base of dimension \( \leq n \) is the inverse image of \( \gamma_0 \) by a \( G \)-morphism of \( P \) in \( B \).

We deduce Theorem 1 from the following theorem which seems to be of independent interest.

**Theorem 2.** Let \( G \) be a connected Lie group and \( \rho \) a finite dimensional representation of \( G \). Let \( n \) and \( p \) be two non-negative integers. Then there exist a principal \( G \)-bundle \( E \) and a \( p \)-form \( \alpha_0 \) of type \( \rho \) on \( E \) such that any \( p \)-form of type \( \rho \) on a principal \( G \)-bundle \( P \) with base of dimension \( \leq n \) is the inverse image of \( \alpha_0 \) by a \( G \)-morphism of \( P \) into \( E \). Moreover, the bundle \( E \) can be chosen to be classifying for dimension \( \leq n \).

**Remarks.** 1. A \( p \)-form (resp. a connection) which possesses the property stated in Theorem 2 (resp. Th. 1) will be called \( n \)-universal.

2. Theorem 2 is also valid with "\( p \)-form" replaced by "exterior \( p \)-form." A universal exterior \( p \)-form is obtained by alternating a universal \( p \)-form.

**Proof of Theorem 1.** We now prove Theorem 1 assuming Theorem 2. Let \( F \) be a differentiable \( G \)-bundle which is \( n \)-universal, and \( \gamma_1 \) any connection on \( F \). On the other hand, let \( E \) be a \( G \)-bundle and \( \alpha_0 \) a 1-form on \( E \) of the adjoint type which is \( n \)-universal for such forms. Consider the \( G \)-bundle \( B = F \times E \), the action of \( G \) on \( B \) being given by \( (f, e)g = (fg, eg), f \in F, e \in E, g \in G \). Let \( q_1: B \to F, q_2: B \to E \) be the canonical projections, which are clearly \( G \)-morphisms. The differential form \( \gamma_0 = q_1^* \gamma_1 + q_2^* \alpha_0 \) is a connection form on \( B \) since \( q_1^* \gamma_1 \) is a connection form and \( q_2^* \alpha_0 \) is a 1-form of the adjoint type. We assert that \( \gamma_0 \) is \( n \)-universal for connections in \( G \)-bundles.
In fact let $P$ be any $G$-bundle with base of dimension $\leq n$ and $\gamma$ any connection on $P$. Since $F$ is a $n$-universal bundle, there exists a $G$-morphism $\Psi_1: P \to F$. Then $\gamma - \Psi_1^*(\gamma_1)$ is a 1-form of the adjoint type since $\gamma$ and $\Psi_1^*(\gamma_1)$ are connection forms on $P$. Let $\Psi_2: P \to E$ be a $G$-morphism such that $\Psi_2^*(\alpha_0) = \gamma - \Psi_1^*(\gamma_1)$. Consider the $G$-morphism $\Phi: P \to B$ defined by $q_1\circ \Phi = \Psi_1; q_2\circ \Phi = \Psi_2$. Then we have

$$
\Phi^*(\gamma_0) = \Phi^*(q_1^*\gamma_1 + q_2^*\alpha_0) = (q_1 \circ \Phi)^*\gamma_1 + (q_2 \circ \Phi)^*\alpha_0 = \Psi_1^*(\gamma_1) + \Psi_2^*(\alpha_0) = \Psi_1^*(\gamma_1) + (\gamma - \Psi_1^*(\gamma_1)) = \gamma.
$$

**Remark.** In the above construction, it is clear that the bundle $B$ is $n$-classifying if the $G$-bundles $E$ and $F$ are $n$-classifying, so that the maps induced on the bases by two $G$-morphisms $P \to B$ are homotopic. Thus the theorem of A. Weil on connections [1] is an immediate consequence of Theorem 1. However, Weil's theorem can be proved in a simpler way; for, all one requires for the proof is that any two given connections $\gamma_1$ and $\gamma_2$ on a bundle $P$ can be obtained as the inverse images of the same connection $\gamma_0$ on a bundle $B$ by morphisms whose projections on the base are homotopic. This problem is considerably simpler as can be seen by taking $B = P \times I$ and $(\gamma_0)(\xi, t) = t\rho^*\gamma_2 + (1 - t)\rho^*\gamma_1$, $\xi \in P$, $t \in I$ where $\rho$ is the projection $P \times I \to P$ ($I$ is the open interval $[-2, 2]$). The inclusions $P \to P \times I$ given by $\xi \mapsto (\xi, 0)$ and $\xi \mapsto (\xi, 1)$ induce $\gamma_1, \gamma_2$ respectively and their projections to the base are clearly homotopic, the homotopy being induced by the identity mapping of $P \times I$ into itself [3, § 6].

**4. The orthogonal case.** In this section, we prove Theorem 2 in the case where $G$ is the real orthogonal group $O(k)$ and $\rho$ is the natural representation in $R^k$. We identify $O(k)$ with the group of $(k, k)$ real matrices $A$ such that $A^tA = I_k$ ($A^t$ being the transpose of $A$) and $R^k$ with $(k, 1)$ real matrices. $\rho$ then corresponds to left multiplication of $(k, 1)$ matrices by $(k, k)$ orthogonal matrices.

Let $W(N, k), N \geq k$, be the Stiefel bundle of $(N, k)$-real matrices $A$ such that $A^tA = I_k$ ([3, § 2]). $O(k)$ acts on $W(N, k)$ by multiplication on the right. If $A \in W(N, k)$ is of the form

$$
\begin{bmatrix}
A_1 \\
\vdots \\
A_N
\end{bmatrix}
$$

where each $A_i$ is a $(1, k)$
matrix, the function $\sigma_i$ on $W(N, k)$ which assigns to each $A$ the $(k, 1)$ matrix $A_i'$ is of type $\rho$. For

$$\sigma_i(As) = \sigma_i \begin{bmatrix} A_1s \\ \vdots \\ A_Ns \end{bmatrix}' = (A_1s)' = A_i' = s^*\sigma_i(A)$$

for $A \in W(N, k)$, $s \in O(k)$.

We now construct a $n$-universal $p$-form of type $\rho$. For the rest of this section, $N$ will denote the integer $(n + 1)n^p + (n + k)$. Let $V_1, \cdots, V_{n+1}$ be $(n + 1)$ copies of $R^n$ and $V_0 = R$. Consider the $O(k)$-bundle

$$E = W(N, k) \times V_0 \times V_1 \times \cdots \times V_{n+1}$$

where the action of $O(k)$ on $E$ is given by

$$(w, v_0, \cdots, v_{n+1})g = (wg, v_0, \cdots, v_{n+1}), w \in W(N, k), v_i \in V_i, g \in O(k).$$

Let $\pi$ (resp. $\pi_i$) denote the projection of $E$ onto $W(N, k)$ (resp. $V_i$). Let further $(x_1^i, \cdots, x^n)$ be the coordinate functions in $V_1$, $i > 0$. For each multi-index $I = (i_1, \cdots, i_r \cdots, i_p)$, $1 \leq i_r \leq n$ and $1 \leq j \leq n + 1$, we shall denote by $\omega_I$ the $p$-form $\pi_i^*$$(dx_{j^1} \otimes \cdots \otimes dx_{j^p})$ on $E$. For convenience of notation, let us choose a bijection $\lambda$ of the set of indices $(I, j)$ with $I = (i_1, \cdots, i_p)$ $1 \leq i_r \leq n$ and $1 \leq j \leq n + 1$, onto the set of integers $[1, (n + 1)n^p]$. Obviously $\tau_I^j = \sigma_{\lambda(I, j)} \circ \pi$ is a function on $E$ with values in $(k, 1)$-matrices of type $\rho$. $\pi_0$ being a real-valued function on $E$, the form $(\pi_0 \cdot \tau_I^j)\omega_I$ is a $(k, 1)$-matrix valued form which is the product of the vector valued function $\pi_0\tau_I^j$ and the real-valued form $\omega_I$. The form

$$\omega = \sum_{I, j} (\pi_0\tau_I^j)\omega_I^j$$

is of type $\rho$. For, clearly $\omega_0$ annihilates any $p$-tuple of vectors one of which is vertical since each $\omega_I^j$ has this property. Moreover if $X_1, \cdots, X_p$ are vectors at $\xi \in P$ and $s \in O(k)$ we have

$$\omega_0(X_1s, \cdots, X_ps) = \sum \pi_0(\xi s)\tau_I^j(\xi s)\omega_I^j(X_1s, \cdots, X_ps)$$

$$= s^*\pi_0(\xi)\sum \tau_I^j(\xi)\omega_I^j(X_1, \cdots, X_p)$$

$$= s^*\omega_0(X_1, \cdots, X_p)$$

where the $X_\xi$s are the vectors at $\xi s$ which are images of $X_i$ by the differential of the map $\xi \rightarrow \xi s$ of $P$ into itself.

We now proceed to prove that $\omega_0$ is $n$-universal for $p$-forms of type $\rho$.

**Proof.** Let $P$ be a $O(k)$-bundle over a base $M$ of dimension $\leq n$ and
$q : P \rightarrow M$ be the projection. Let $(U_i)$ be a covering of $M$ by relatively compact open sets such that

i) each $\bar{U}_i$ is contained in a coordinate cell

ii) the $U_i$'s can be divided into $(n + 1)$ classes $E_j$ in such a way that no two $U_i$'s of the same class intersect [4, p. 61].

Let $W_i$ be a shrinking of this covering, i.e., an open covering $W_i$ such that $\bar{W}_i \subset U_i$. Let $D_j \{ j = 1, \cdots, (n + 1) \}$ be the union of the open sets $q^{-1}(W_i)$ for those $i$'s for which $U_i$ belongs to $E_j$. Let $\xi_j$ be a partition of unity with respect to this covering, consisting of non-negative differentiable functions $\xi_j$ invariant under the action of $G$ with support of $\xi_j \subset D_j$ and $\sum \xi_j = 1$.

Let $\alpha$ be a $p$-form of type $\rho$ on $P$. It is clear that there exist functions $(f^1, \cdots, f^n)$ on $M$ whose restrictions to each $W_i$, for those $i$'s for which $U_i$ belongs to $E_j$, form a coordinate system on $W_i$. Since $\alpha$ is of type $\rho$, $\alpha$ can be expressed in $D_j$ in the form $\sum \alpha^I q^I (df^I)$, where $\alpha^I$ are functions of type $\rho$ on $D_j$ and $df^I = df^I_1 \otimes \cdots \otimes df^I_n$. Then it is easy to see that $\alpha = \sum \beta^I q^I (df^I)$, where $\beta^I = \xi_j \alpha^I$ are now differentiable functions on $P$ of type $\rho$. Let $h$ be a strictly positive invariant differentiable function on $P$ such that $h(\xi)^2 > 2 \| \beta^I(\xi) \beta^I(\xi)' \|$ for every $\xi \in P$, where $\| \|$ denotes the norm as a linear operator. (The existence of such an $h$ follows for instance from the fact that $\| \beta^I(\xi) \beta^I(\xi)' \|$ is an invariant function on $P$). We have

$$\alpha = \sum h \eta^I q^I (df^I)$$

where $\eta^I = \frac{1}{h} \beta^I$. Obviously

$$\| \eta^I(\xi) \eta^I(\xi)' \| = \frac{1}{h(\xi)^2} \| \sum \beta^I(\xi) \beta^I(\xi)' \| \leq \frac{1}{2}.$$ 

Therefore $R(\xi) = I_k - \sum \eta^I(\xi) \eta^I(\xi)'$ is a function on $P$ with values in positive definite matrices. Moreover, for $s \in O(k)$ and $\xi \in P$,

$$R(\xi s) = s^{-1} R(\xi)s.$$ 

For,

$$R(\xi s) = I_k - \sum \eta^I(\xi s) \eta^I(\xi s)'$$

$$= I_k - \sum s^{-1} \eta^I(\xi) \eta^I(\xi)' (s^{-1})'$$

$$= s^{-1} (I_k - \sum \eta^I(\xi) \eta^I(\xi)') s$$

$$= s^{-1} R(\xi)s.$$ 

Let $S(\xi)$ be the differentiable positive matrix-valued function on $P$ such that
$S(\xi)^2 = R(\xi)$. It is clear from the uniqueness of the positive square root of a positive definite matrix that $S(\xi) = s^{-1} S(\xi) s$ for $\xi \in P$, $s \in O(k)$.

Let $\psi: P \to W(n + k, k)$ be a $G$-morphism, the existence of which is assured by the universal bundle theorem [6, §19]. Consider the matrix

$$\psi_1(\xi) = \begin{pmatrix}
\eta_1(\xi)', \\
\vdots \\
\eta_n(\xi)', \\
\eta_{(n+1)p}(\xi)', \\
\psi(\xi) S(\xi)
\end{pmatrix}$$

where $\eta_i = \eta_i'$ with $\lambda(I, J) = i$. Each $\eta_i'$ is a $(1, k)$ matrix and $\psi(\xi) S(\xi)$ is a $(n + k, k)$ matrix so that $\psi_1(\xi)$ is a $((n + 1)p + n + k, k)$ matrix. The map $\xi \mapsto \psi_1(\xi)$ is a map of $P$ into $W(N, k)$. For,

$$\psi_1'(\xi) \psi_1(\xi) = \sum_{i=1}^{(n+1)p} \eta_i(\xi) \eta_i(\xi)' + S(\xi)' \psi(\xi)' \psi(\xi) S(\xi)$$

$$= \sum_{(i,j)} \eta_i'(\xi) (\eta_j'(\xi))' + S(\xi)^2$$

$$= I_k - R(\xi) + S(\xi)^2.$$ 

Hence $\psi_1'(\xi) \psi_1(\xi) = I_k$.

Moreover $\psi_1: P \to W(N, k)$ is a $G$-morphism. In fact,

$$\psi_1(\xi s) = \begin{pmatrix}
\eta_1(\xi s)', \\
\vdots \\
\eta_n(\xi s)', \\
\eta_{(n+1)p}(\xi s)', \\
\psi(\xi s) S(\xi s)
\end{pmatrix} = \begin{pmatrix}
(s^{-1} \eta_1(\xi))', \\
\vdots \\
(s^{-1} \eta_{(n+1)p}(\xi))', \\
\psi(\xi)s \cdot s^{-1}S(\xi)s
\end{pmatrix} \psi_1(\xi s).$$

Finally, we construct a $G$-morphism $\Phi$ of $P$ into $E$ such that $\Phi^* \alpha_0 = \alpha$ ($E$ and $\alpha_0$ are the bundle and $p$-form constructed in the beginning of this section). $\Phi: P \to E$ is defined by $\pi_0 \circ \Phi = h$, $\pi_j \circ \Phi = (f_j^1 \circ q, \cdots , f_j^n \circ q)'$, $(j=1, \cdots , n+1)$ and $\pi \circ \Phi = \psi_1$. $\Phi$ is a $G$-morphism since $\psi_1$ is so and $h$, $f_j^i \circ \pi$ are invariant functions. We then have

$$\Phi^* (\alpha_0) = \Phi^* (\sum \pi_0 \tau f_{0j})$$

$$= \sum (\pi_0 \circ \Phi) (\tau f_{0j} \circ \Phi) \Phi^* (\omega f^I)$$
\[= \sum h(\sigma_{\lambda(I,j)} \circ \pi \circ \Phi) \Phi^*(\omega_I)\]
\[= \sum h\eta_{(I,j)} \Phi^* \pi^j_*(dx_1^{i_1} \otimes \cdots \otimes dx_1^{i_p})\]

where \(I = (i_1, \ldots, i_p)\) with \(1 \leq i_p \leq n\). Hence
\[\Phi^*(\alpha_0) = \sum (h\eta^j_I \circ q) \otimes \cdots \otimes (dx_1^{i_p} \circ q) = \alpha.\]

5. The case of a compact group. In this section we prove Theorem 2 with \(G\) compact and \(\rho\) any \(k\)-dimensional representation of \(G\). Since every representation of \(G\) is equivalent to an orthogonal representation, we may assume that \(\rho = j \circ f\) where \(f\) is a homomorphism \(G \rightarrow O(k)\) and \(j\) is the natural representation of \(O(k)\) in \(\mathbf{R}^k\). Let \(E_1\) be a \(O(k)\)-bundle together with an \(n\)-universal \(p\)-form of type \(j\) (§ 4). Let \(F\) be a \(n\)-universal \(G\)-bundle. We let \(G\) act on \(F \times E_1\) by \((v, e_1, g) = (vg, e_1f(g)), v \in F, e_1 \in E_1, g \in G\). This makes \(F \times E_1\) a \(G\)-bundle \(E_2\). Let \(q_1\) and \(q_2\) be the projections of \(F \times E_1\) onto \(F\) and \(E_1\) respectively. The \(p\)-form \(\alpha_0 = q_2^* \alpha_1\) is of type \(\rho\) since \(q_2: E \rightarrow E_1\) is a \(f\)-morphism and \(\alpha_1\) of type \(j\). We now prove that \(\alpha_0\) is an \(n\)-universal \(p\)-form of type \(\rho\). In fact, let \(P\) be a \(G\)-bundle over a base of dimension \(\leq n\) and \(\alpha\) a \(p\)-form of type \(\rho\) on \(P\). Then there exists a \(O(k)\)-morphism \(\Phi_2\) of \(T_f(P)\) into \(E_1\) such that \(\Phi_2^*(\alpha_1) = T_f(\alpha)\). (For the definition of \(T_f\) see § 2). On the other hand, \(P\) admits a \(G\)-morphism \(\Phi_1\) into \(F\), since \(F\) is \(n\)-universal. Let \(\Phi: P \rightarrow E\) be the map defined by \(q_1 \circ \Phi = \Phi_1, q_2 \circ \Phi = \Phi_2 \circ i_f\) (\(i_f\) is the canonical map \(P \rightarrow T_f(P)\), see § 2). \(\Phi\) is a \(G\)-morphism. For, if \(\xi \in P, s \in G\), we have
\[\Phi(\xi s) = (\Phi_1(\xi s), \Phi_2 \circ i_f(\xi s))\]
\[= (\Phi_1(\xi) s, \Phi_2(i_f(\xi) f(s)))\]
\[= \Phi(\xi) s.\]

Now
\[\Phi^*(\alpha_0) = \Phi^*(q_2^* \alpha_1) = (q_2 \circ \Phi)^* \alpha_1 = i_f^* \Phi_2^* \alpha_1 = i_f^* T_f \alpha = \alpha.\]

This completes the proof in the compact case.

6. Proof of Theorem 2. The general case. Let \(G\) be a connected Lie group and \(K\) a maximal compact subgroup of \(G\). We denote by \(f\) the inclusion map \(K \rightarrow G\). We seek to construct a \(n\)-universal \(p\)-form of type \(\rho\), where \(\rho\) is any finite dimensional representation of \(G\). From § 5, there exists a \(n\)-universal \(p\)-form \(\alpha_1\) of type \((\rho \circ f)\) on a \(K\)-bundle \(E_1\). Then the \(p\)-form \(\alpha_0 = T_f(\alpha_1)\) on the \(G\)-bundle \(E = T_f(E_1)\) is \(n\)-universal for \(p\)-forms of
type $\rho$. In fact, let $P$ be a $G$-bundle over a base of dimension $\leq n$ and $\alpha$ a $p$-form of type $\rho$ on $P$. It is well known that there exists a $K$-bundle $P_1$ such that $T_f(P_1)$ is $G$-isomorphic to $P$ (reduction of structure group to $K$, see [6, §12]). We identify $P$ and $T_f(P_1)$ by such an isomorphism. Consider the form $i_f^*(\alpha)$ on $P_1$ which is of type $\rho \circ f$. Now let $\Phi_1: P_1 \to E_1$ be a $K$-morphism such that $\Phi_1^*(\alpha_1) = i_f^*(\alpha)$. Consider the $G$-morphism $\Phi = T_f(\Phi_1)$ of $P$ into $E$. Then we have

$$
\Phi^*(\alpha) = (T_f(\Phi_1))^*\alpha_0 = (T_f(\Phi_1))^*T_f(\alpha_1)
= T_f(\Phi_1^*\alpha_1) = T_f(i_f^*\alpha)
= \alpha.
$$

It is clear, referring to §§ 4, 5, 6, that the bundle $E$ can be chosen to be $n$-classifying. This completes the proof of Theorem 2.

**Remark.** Theorems 1 and 2 hold even when $G$ is a Lie group with a finite number of connected components; our proofs continue to be valid in this case.

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