A DUALITY FOR SPIN VERLINDE SPACES AND PRYM
THETA FUNCTIONS

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Abstract

The paper proves canonical isomorphisms between Spin Verlinde spaces, that is, spaces of global sections of a determinant line bundle over the moduli space of semistable Spin\(_n\)-bundles over a smooth projective curve \(C\), and the dual spaces of theta functions over Prym varieties of unramified double covers of \(C\).

1. Introduction

To any smooth, projective curve \(C\), one classically associates a collection of principally polarized abelian varieties: the Jacobian \(JC\), parametrizing degree zero line bundles, as well as, for any unramified double cover \(C\rightarrow C\), a Prym variety \(P\). The projective geometry of the configuration \(JC\cup\bigcup P\) has been much studied [1, 8, 14] and encodes, for example, the Schottky–Jung identities among theta-constants [14].

Less classically, one can consider the moduli space \(\mathcal{M}(G)\) of semistable principal \(G\)-bundles over the curve \(C\), where \(G\) is a simple and simply-connected algebraic group. For some ample line bundle \(L\) over \(\mathcal{M}(G)\), the vector space of global sections \(H^0(\mathcal{M}(G), L)\) has been identified to a space of conformal blocks arising in conformal field theory (see for example [22] for a survey or [11] for a proof), which made the computation of its dimension possible. This is the celebrated Verlinde formula.

In this paper we are interested in dualities between Verlinde spaces \(H^0(\mathcal{M}(G), L)\) and spaces of abelian theta functions, that is, sections of some multiple of a principal polarization over \(JC\) or, more generally, \(JC\cup\bigcup P\). Such dualities were first proved for the structure group \(G = SL_2\) and the line bundles \(L^l\) for \(l = 1, 2, 4\) [1, 2, 17] or, more generally, \(G = SL_n\) and the line bundle \(L\) [4], where \(L\) is the ample generator of the Picard group of \(\mathcal{M}(SL_n)\).

In [15, 16], W. M. Oxbury constructs linear maps between a Verlinde space for the complex spin group, that is, \(G = Spin_n\), and a space of abelian theta functions over \(JC\cup\bigcup P\). Our main theorem states that these linear maps are actually isomorphisms. More precisely we show the following theorem.

Theorem 1.1. For any curve \(C\) and any integer \(m \geq 1\), we have canonical isomorphisms:

\[
\sum s_\eta^\pm \cdot \sum_{\eta \in JC[2]} H^0(P^{ev}_\eta, (2m + 1) \Xi_\eta^\pm) \xrightarrow{\sim} H^0(\mathcal{M}(Spin_{2m+1}), \Theta(C^{2m+1})),
\]

\[
\sum s_\eta^\pm \cdot \sum_{\eta \in JC[2]} [H^0(P^{ev}_\eta, (2m) \Xi_\eta^\pm) \oplus H^0(P^{odd}_\eta, (2m) \Xi_\eta^\pm)] \xrightarrow{\sim} H^0(\mathcal{M}(Spin_{2m}), \Theta(C^{2m})).
\]
We refer to Sections 2 and 3 for the rather technical details encoded in the notation.

We note that Theorem 1.1 for the group Spin$_g$ coincides with the above-mentioned duality for the Verlinde space $H^0(\mathcal{M}(\text{SL}_2), \mathcal{L}^t)$ under the exceptional isomorphism Spin$_g = \text{SL}_2$. In this case Theorem 1.1 was proved in [21] for all curves and in [17] for the generic curve.

The proof of the theorem essentially goes as follows. First, we construct maps from products of Prym varieties, called $\text{SCH}_n$, into the moduli space $(\text{Spin}_n)^t$. This is done by induction on $n$, exploiting the exceptional isomorphisms of Spin$_n$ with algebraic groups for small values of $n$. For example, $\text{SCH}_n$ is isomorphic to the union $\text{JC}_{e} \cap \mathcal{P} \eta \mathcal{P}$. Next, we observe that the divisor in the product $(\text{Spin}_n)^t \cap (\text{JC}_{e} \cap \mathcal{P} \eta \mathcal{P})$, which induces the pairing of the theorem, decomposes when restricted to the variety $\text{SCH}_n$. Hence we obtain a factorization of the linear map of Theorem 1.1. As our auxiliary variety $\text{SCH}_n$ is constructed from abelian varieties, we can use theorems on multiplication of theta functions (Propositions 4.1 and 4.2) and dualities for second order theta functions (Prym–Wirtinger duality, Proposition 5.1) in order to show, again inductively, that all maps of the factorization are injective. By the Verlinde formula, this will be enough to deduce the theorem.

It is legitimate to ask whether the isomorphism of Theorem 1.1 is a particular case of more general duality. Indeed, in the case $G = \text{SL}_n$, one constructs ([3, Section 8] and [7]), via canonical theta divisors, pairings among $\text{SL}_n$- and $\text{GL}_n$- Verlinde spaces, the so-called ‘strange duality’ or ‘rank-level duality’. Some evidence (see [16, Remark 3.4] and [18]) suggests that this ‘strange duality’ phenomenon should also occur for Spin$_n$-bundles.

As we have remarked above, the duality was proved in [21] in the case of $\text{SL}_2$ and the proof was later generalized by the first author to the case of Spin groups. In order to minimize the number of papers, we decided to publish our results together.

2. Notation and preliminaries

2.1. Prym varieties

Given a non-zero $\zeta \in \text{JC}[2]$, the group of 2-torsion points of the Jacobian $\text{JC}$, we consider its associated two-sheeted étale cover $\pi^\zeta: \tilde{\text{C}} \longrightarrow C$ and the norm map $\text{Nm}: \text{Pic}(\tilde{\text{C}}) \longrightarrow \text{Pic}(C)$. Mumford [14] introduced the following (isomorphic) subvarieties of $\text{Pic}(\tilde{\text{C}})$, called Prym varieties:

$$\text{Nm}^{-1}(\zeta) = P^{\zeta} \cup P^{'\zeta}, \quad \text{Nm}^{-1}(\omega) = P^{\zeta,\omega} \cup P^{\zeta,\text{odd}},$$

(2.1)

where $P^{\zeta}$ is the connected component containing the origin $\zeta \in \text{JC}$ and $P^{'\zeta}$ is the other component. The line bundle $\omega$ is the canonical line bundle on $C$. For a description of $\text{Nm}^{-1}(\omega)$, see Example 2.1. The Galois involution $\sigma_\zeta$ of $\tilde{\text{C}}$ acts by pull-back on the four Prym varieties. We shall also denote the corresponding involutions by $\sigma_\zeta$. On $P^{\zeta} \cup P^{'\zeta}$, we have $\sigma_\zeta(L) = L^{-1}$ and on $P^{\zeta,\omega} \cup P^{\zeta,\text{odd}}$, we have $\sigma_\zeta(\zeta) = \omega \zeta^{-1}$. In this paper we will also use

$$P^{'\zeta} = \text{one connected component of } \text{Nm}^{-1}(\zeta).$$

In order to have a consistent notation, we put $P^{\zeta} = \text{JC}$ and $P^{'\zeta} = \emptyset$, $P^{\zeta,\omega} = \text{Pic}^{\omega}(C)$ and $P^{\zeta,\text{odd}} = \emptyset$. On the finite group $\text{JC}[2]$ we have the skew-symmetric Weil pairing
which we denote by \( \langle \eta, \zeta \rangle \in \{ \pm 1 \} \). The group of 2-torsion points \( P_2^\alpha[2] \) of the Prym \( P_\zeta^\alpha \) is isomorphic to the annihilator (with respect to the Weil pairing) of \( \zeta \in JC[2] \), that is,

\[
P_2^\alpha[2] \xrightarrow{\epsilon^\alpha} \langle \zeta \rangle^* / \langle \zeta \rangle.
\]

(2.2)

Consider \( \eta \in JC[2] \) with \( \langle \eta, \zeta \rangle = 1 \). We also denote by \( \eta \in P_2^\alpha[2] \) the corresponding element under isomorphism (2.2) and by \( T_\eta \) the translation by \( \eta \) acting on \( P_\zeta^\alpha \) as well as on the other three Prym varieties (2.1).

The Prym variety \( P_\zeta^\alpha \) comes equipped with a naturally defined reduced Riemann theta divisor \( \Xi \), whose set-theoretical support equals

\[
\Xi := \{ \xi \in P_\zeta^\alpha | h^0(\tilde{C}, \xi) > 0 \}.
\]

(2.3)

A translate by a theta-characteristic (respectively a 2-torsion point) of \( \Xi \) gives (non-canonically) a symmetric theta divisor on \( P_\zeta^\alpha \) and \( P_\zeta^\beta \) (respectively \( P_\zeta^{\alpha \beta} \)) (see also Section 2.2), which we also denote by \( \Xi \).

Given \( \eta \in P_2^\alpha[2] \), we define the line bundle on \( P_\zeta^\alpha \):

\[
\mathcal{L}_\eta^\alpha := \mathcal{L}(\xi + T_\eta \Xi).
\]

Here \( P_\zeta^\alpha \) stands for any of the four varieties in (2.1). We note that \( \Xi \) is only canonically defined on \( P_\zeta^\alpha \) (2.3), but the line bundle \( \mathcal{L}_\eta^\alpha \) over any \( P_\zeta^\alpha \) does not depend on the choice of \( \Xi \). Moreover, the involutions \( \sigma_\zeta \) of \( P_\zeta^\alpha \) lift canonically to a linear involution \( \sigma_\eta^\alpha \) of \( H^0(P_\zeta^\alpha, \mathcal{L}_\eta^\alpha) \). The subscript + (respectively −) will denote the + (respectively −) eigenspace of \( \sigma_\eta^\alpha \).

2.2. Theta-characteristics

In this section we recall some basic results on theta-characteristics on Prym varieties. Let \( \mathcal{H}(C) \) be the set of theta-characteristics on \( C \), that is,

\[
\mathcal{H}(C) = \{ \kappa \in \text{Pic}^e(C) | \kappa^\alpha \cong \omega \},
\]

which comes equipped with a parity map \( \epsilon : \mathcal{H}(C) \longrightarrow \{ \pm 1 \} ; \epsilon(\kappa) = (-1)^{h^0(\omega)} \). There is a one-to-one correspondence between \( \kappa \in \mathcal{H}(C) \) and functions \( \tilde{\kappa} : JC[2] \longrightarrow \{ \pm 1 \} \) which satisfy, for all \( \eta, \zeta \in JC[2] \),

\[
\tilde{\kappa}(\eta + \zeta) = \tilde{\kappa}(\eta) \tilde{\kappa}(\zeta) \langle \eta, \zeta \rangle.
\]

(2.4)

The correspondence associates to \( \kappa \in \mathcal{H}(C) \) the function \( \tilde{\kappa}(\eta) = (-1)^{h^0(\kappa \otimes \eta) + h^0(\kappa) \omega} \). The group \( JC[2] \) acts on \( \mathcal{H}(C) \) by tensor product and we have \( \epsilon(\eta \kappa) = \epsilon(\kappa) \tilde{\kappa}(\eta) \). We define the set \( \mathcal{H}(P_2^\alpha) \) of theta-characteristics on \( P_2^\alpha \) to be the set of functions \( \tilde{\kappa} : P_2^\alpha \cong \langle \zeta \rangle^* / \langle \zeta \rangle \longrightarrow \{ \pm 1 \} \) satisfying (2.4). Then we have, using the correspondence between \( \kappa \) and \( \tilde{\kappa} \),

\[
\mathcal{H}(P_2^\alpha) = \{ \kappa \in \mathcal{H}(C) | \tilde{\kappa}(\zeta) = 1 \} / \langle \zeta \rangle.
\]

Note that we have an equivalence \( \kappa \in \mathcal{H}(P_2^\alpha) \iff \pi_\#^* \kappa \in P_2^\alpha \). Given \( \kappa \in \mathcal{H}(P_2^\alpha) \), we consider the symmetric theta divisor \( T_\zeta \Xi \) on \( P_\zeta^\alpha \). We shall write \( \kappa \) instead of \( \pi_\#^* \kappa \). Let \( i \) be the unique isomorphism of \( \sigma_\#^\alpha s_\zeta \in \mathcal{E}(T_\zeta \Xi) \) with \( \mathcal{E}(T_\zeta \Xi) \) which induces the identity at the origin, then we have

\[
i(\sigma_\#^\alpha s_\zeta) = \epsilon(\kappa) s_\zeta
\]

(2.5)

where \( s_\zeta \) is the unique (up to a multiplicative constant) global section of \( \mathcal{E}(T_\zeta \Xi) \). We have similar results for the component \( P_\zeta^\beta \). We define

\[
\mathcal{H}(P_\zeta^\beta) = \{ \kappa \in \mathcal{H}(C) | \tilde{\kappa}(\zeta) = -1 \} / \langle \zeta \rangle = \{ \kappa \in \mathcal{H}(C) | \pi_\#^* \kappa \in P_\zeta^{\alpha \beta} \}.
\]
2.3. Mumford’s parity conservation theorem and Pfaffian line bundles

We recall here Mumford’s parity conservation theorem [13] which says that the parity of \( h^0 \) of vector bundles on \( C \) is preserved under deformation, if there exists a non-degenerate quadratic form on the family with values in \( \omega \).

**Example 2.1.** For all \( \xi \in P^\omega_\xi \cup P^\text{odd}_\xi \), the rank 2 vector bundle \( E = \pi_\xi \otimes \xi \) carries a quadratic form given by the Norm homomorphism \( \text{Nm}: \pi_\xi \otimes \xi \to \omega \). This explains the notation \( P^\omega_\xi \) and \( P^\text{odd}_\xi \), since the two components of \( \text{Nm}^{-1}(\omega) \) may be distinguished by the property that \( h^0(\pi_\xi \otimes \xi) \) is even for \( \xi \in P^\omega_\xi \) and odd for \( \xi \in P^\text{odd}_\xi \).

**Example 2.2.** Consider \( \eta, \zeta \in JC[2] \) with \( \langle \eta, \zeta \rangle = 1 \). For all \( \xi \in P^\omega_\eta \cup P^\text{odd}_\eta \) and all \( L \in P^\omega_\zeta \cup P^\text{odd}_\zeta \), the rank 4 vector bundle \( \pi_\eta \otimes \eta \otimes \pi_\zeta L \) has a quadratic form, which is obtained by multiplying the \( \omega \)-valued form on \( \pi_\eta \otimes \eta \) and the \( \zeta \)-valued form on \( \pi_\zeta L \) (see Example 2.1).

**Example 2.3.** Consider the rank 2\( n \) vector bundle \( \pi_\eta \otimes \eta \otimes F \), where \( \pi_\eta \otimes \eta \) is as in Example 2.1) and \( F \) is an orthogonal vector bundle, that is, equipped with an \( \zeta \)-valued quadratic form. Multiplying the two forms gives the required \( \omega \)-valued form on \( \pi_\eta \otimes \eta \otimes F \).

Given a family \( \mathcal{E} \) of vector bundles over \( C \) with an \( \omega \)-valued quadratic form and parametrized by an integral variety \( S \), one can construct a Pfaffian line bundle \( \text{PF}(\mathcal{E}) \) over \( S \) and a Pfaffian divisor \( \text{divPF}(\mathcal{E}) \), which are square roots of the determinant line bundle \( \text{DET}(\mathcal{E}) \) and determinant divisor \( \text{divDET}(\mathcal{E}) \), that is,

\[
\text{DET}(\mathcal{E}) = \text{PF}(\mathcal{E}) \otimes 2, \quad 2\text{divPF}(\mathcal{E}) = \text{divDET}(\mathcal{E}).
\]

For the construction of \( \text{PF}(\mathcal{E}) \) and \( \text{divPF}(\mathcal{E}) \) we refer to [11, Proposition 7.9]. The support of the Cartier divisor \( \text{divPF}(\mathcal{E}) \) equals

\[
\text{supp} \text{divPF}(\mathcal{E}) = \{ s \in S \mid h^0(\mathcal{E}_s) > 0 \}.
\]

Thus, the Pfaffian divisor associated to the family \( \pi_\xi \otimes \xi \) with \( \xi \in P^\omega_\xi \) (Example 2.1) is the Riemann theta divisor \( \Xi_\xi \) in (2.3). The Pfaffian divisors associated to the families of Example 2.2) (respectively Example 2.3) will appear in the proof of Proposition 5.1 (respectively in the construction of the Prym–Spin duality, see Section 3).

3. Prym varieties and the moduli of \( \text{Spin}_n \)-bundles

In this section we recall the definition of the linear maps of Theorem 1.1 (see [15, 16]). Let \( \mathcal{SC}_n \) be the special Clifford group of a non-degenerate quadratic form of dimension \( n \) and \( \mathcal{M}(\mathcal{SC}_n) \) be the moduli space of semistable principal \( \mathcal{SC}_n \)-bundles. The spinor norm induces a surjective morphism \( \text{Nm}: \mathcal{M}(\mathcal{SC}_n) \to \text{Pic}(C) \) and we define

\[
\mathcal{M}^+(\text{Spin}_n) = \mathcal{M}(\text{Spin}_n) = \text{Nm}^{-1}(\mathcal{E}), \quad \mathcal{M}^-(\text{Spin}_n) = \text{Nm}^{-1}(\mathcal{E}(p)),
\]

where \( p \) is a fixed point on \( C \). In Section 6 we shall give examples of the moduli spaces \( \mathcal{M}^+(\text{Spin}_n) \) for low values of \( n \). We denote by \( V \cong \mathbb{C}^n \) the standard orthogonal
representation of SC\(_n\) and by S (if \(n\) is odd) and S\(^\pm\) (if \(n\) is even) the spinor and half-spinor representations. Given a Clifford or Spin-bundle \(A\), we shall denote by \(A(V)\), \(A(S)\), \(A(S^\pm)\) the induced vector bundles.

There is a natural action of \(JC[2]\) on the moduli spaces \(H^\pm(S_n)\) and we recall that there are natural Galois covers

\[
H^\pm(S_n) \xrightarrow{JC[2]} H^\pm(SO_n), \quad A \mapsto A(V). \tag{3.1}
\]

The Galois action of \(JC[2]\) on the moduli \(H^\pm(S_n)\) is given by \(A \mapsto \alpha \cdot A\) for \(\alpha \in JC[2]\) with

\[
(\alpha \cdot A)(V) = A(V), \quad (\alpha \cdot A)(S^\pm) = A(S^\pm) \otimes \alpha. \tag{3.2}
\]

The two components \(H^\pm(SO_n)\) of the moduli of semistable orthogonal bundles are distinguished by the second Stiefel–Whitney class of the bundles. For some general facts on orthogonal bundles, see [20].

To construct the linear maps of Theorem 1.1, it will be enough to exhibit effective Cartier divisors in the right linear systems

\[
D_{2m+1, q}^\pm \subset P^q \times H^\pm(Spin_{2m+1}), \quad D_{2m, q}^\pm \subset (P^q \cup P^q) \times H^\pm(Spin_{2m}).
\]

We observed in Example 2.3 that the family \(\mathcal{E} := \{\pi, \xi \otimes A(V)\}\) carries an \(\omega\)-valued form, so we define \(D_{q, \eta} := \text{divPF}(\mathcal{E})\). By (2.6) we have (for example, for \(n\) odd)

\[
\text{supp} D_{2m+1, q}^\pm = \{((\xi, A) \in P^q \times H^\pm(Spin_{2m+1}) | h^0(C, \pi, \xi \otimes A(V)) > 0)\}. \tag{3.3}
\]

We shall use the abbreviated notation \(D_q\) if there is no ambiguity. We notice that the divisor \(D_q\) does not descend to a Cartier divisor on \(P_q \times H^\pm(SO_n)\). We denote by \(s_n^\pm\) their associated linear maps. The details of this construction are worked out in [16, Section 6].

Given a pair of semistable orthogonal vector bundles \((E, E')\) in \(H^\pm(SO_n) \times H^\pm(SO_n)\) we can associate its orthogonal direct sum \(E \oplus E' \in H^\pm(SO_{n+m})\), which gives rise to a morphism \(H^\pm(SO_n) \times H^\pm(SO_n) \xrightarrow{\text{nat.}} H^\pm(SO_{n+m})\). In the next lemma we will show that this morphism lifts to the moduli spaces of Spin-bundles.

**Lemma 3.1.** For any \(n, m \geq 1\), there are natural morphisms:

\[
H^\pm(Spin_{2n}) \times H^\pm(Spin_{2m}) \xrightarrow{i} H^\pm(Spin_{2n+2m}), \tag{3.4}
\]

\[
H^\pm(Spin_{2n+1}) \times H^\pm(Spin_{2m}) \xrightarrow{i} H^\pm(Spin_{2n+2m+1}). \tag{3.5}
\]

which are lifts via (3.1) of the direct sum morphisms for orthogonal vector bundles. Given a pair \((A, A')\) of Spin-bundles, then its sum \(A + A' := i(A, A')\) has the following properties.

1. The two associated half-spinor vector bundles are

\[
(A + A')(S^+) = A(S^+) \oplus A(S^-) \oplus A(S' \oplus A(S'),
\]

\[
(A + A')(S^-) = A(S') \oplus A(S') \oplus A(S^-) \oplus A(S') .
\]

2. The associated spinor vector bundle is

\[
(A + A')(S) = A(S) \oplus (A(S') \oplus A(S^-)).
\]
Moreover the pull-back by $i$ of the determinant line bundle decomposes ($l = 2n$ or $2n+1$)
\[ r^*\Theta(C^{l+2m}) = \Theta(C') \boxtimes \Theta(C^{2m}). \] (3.6)

**Proof.** The natural homomorphism of algebraic groups $SC_i \times Spin_{2m} \longrightarrow SC_{l+2m}$ induces a morphism at the level of Clifford- (and Spin-) bundles. By [20, Proposition 4.2] and [15, Lemma 1.2] semistability is preserved under this operation. Thus, taking into account that the spinor norm is preserved under the above group homomorphism and that $\mathcal{M}^\pm(Spin_n)$ is a coarse moduli space, we get the above-claimed morphisms. The expressions of the associated (half-)spinor bundles are easily deduced from the definitions of the (half-)spinor representations of the Spin groups.

\[ \square \]

4. **Multiplication of theta functions**

In this section we recall some facts on theta functions over an abelian variety. Let $(X, \Theta)$ be a principally polarized abelian variety, where $\Theta$ is a symmetric divisor representing the polarization. The subscript $\pm$ denotes the $\pm$-eigenspaces of $H^0(X, m\Theta)$ under the canonical involution of $X$. Recall that $H^0(X, 2\Theta)_\pm = H^0(X, 2\Theta)$. We will need the following facts on multiplication maps.

**Proposition 4.1.** If $l \geq 2$ and $m \geq 3$, then the multiplication maps
1. $H^0(X, l\Theta) \otimes H^0(X, m\Theta) \longrightarrow H^0(X, (l+m)\Theta)$;
2. $H^0(X, 2\Theta)_\pm \otimes H^0(X, m\Theta) \longrightarrow H^0(X, (2l+m)\Theta)$

are surjective.

**Proof.** For a proof of (1), we refer, for example, to [9]. For a proof of (2), see [10, Proposition 1.4.4].

The symmetric theta divisor $\Theta$ allows us to identify $X$ with its dual variety $\hat{X}$. Let $m$ be the isogeny
\[ m : X \times X \longrightarrow X \times X \]
\[ (x, y) \longmapsto (x + y, x - y). \] (4.1)

Then it is well known that for any $x \in X[2] = \hat{X}[2]$, we have
\[ m^*\mathcal{C}_x(2\Theta \otimes x) \boxtimes \mathcal{C}_x(2\Theta \otimes x) = \mathcal{C}_x(4\Theta) \boxtimes \mathcal{C}_x(4\Theta). \]

We take global sections and take the sum over all $x \in X[2]$ to get the direct sum decomposition
\[ m^* : \sum_{x \in X[2]} H^0(X, 2\Theta \otimes x) \otimes H^0(X, 2\Theta \otimes x) \longrightarrow H^0(X, 4\Theta) \otimes H^0(X, 4\Theta). \] (4.2)

Since $m$ is equivariant for the canonical involution of $X \times X$, we obtain the following decomposition into $\pm$-eigenspaces:
\[ \sum_{x \in X[2]} H^0 \otimes H^0 \otimes H^0 \otimes H^0 = [H^0(X, 4\Theta) \otimes H^0(X, 4\Theta)], \] (4.3)
\[ \sum_{x \in X[2]} H^0 \otimes H^0 \otimes H^0 \otimes H^0 = [H^0(X, 4\Theta) \otimes H^0(X, 4\Theta)]. \] (4.4)
In particular, if we restrict $m$ to the diagonal $X \subseteq X \times X$ we get the direct sum decomposition (in this case we get the duplication map $m_{|X} = [2]: X \rightarrow X; x \mapsto 2x$):

$$[2]^*: \bigoplus_{x \in X[2]} H^n(X, 2\Theta \otimes x) \rightarrow H^n(X, 8\Theta).$$

(4.5)

In the next proposition we will need a more general version, for all $n \geq 1$

$$[2]^*: \bigoplus_{x \in X[2]} H^n(X, n\Theta \otimes x) \rightarrow H^n(X, 4n\Theta).$$

(4.6)

Via the Weil pairing we can identify $X[2]$ with the group of characters $\text{Hom}(X[2], \mathbb{C}^*)$: $x \mapsto \langle x, \cdot \rangle$. The natural action of the group $X[2]$ on $X$ and $X \times X$ lifts canonically to a linear action on the right-hand side spaces of (4.2), (4.5) and (4.6). Then the direct sum decompositions of (4.2), (4.5) and (4.6) are precisely the character space decompositions of this group action.

**Proposition 4.2.** The following multiplication maps are surjective.

1. $\sum_{x \in X[2]} H^n(X, 2\Theta \otimes x) \otimes H^n(X, \Theta \otimes x) \rightarrow H^n(X, 3\Theta)$. 
2. $\sum_{x \in X[2]} H^n(X, 2\Theta \otimes x), \otimes H^n(X, 2\Theta \otimes x) \rightarrow H^n(X, 4\Theta)$. 

Proof. First let us prove (1). By Proposition 4.1(1) ($l = 4, m = 8$), the multiplication map

$$H^n(X, 4\Theta) \otimes H^n(X, 8\Theta) \rightarrow H^n(X, 12\Theta)$$

(4.7)

is surjective. Consider the character space decomposition (4.6) of the two spaces. Since the above tensor product is compatible with the linear action of the group $X[2]$, we get a surjective map between the two character spaces of (4.7) corresponding to the zero character, that is,

$$\sum_{x \in X[2]} H^n(X, 2\Theta \otimes x) \otimes H^n(X, \Theta \otimes x) \rightarrow H^n(X, 3\Theta).$$

Considering $\pm$ eigenspaces proves assertion (1). By Proposition 4.1(2) ($l = 4, m = 8$), the multiplication map

$$H^n(X, 8\Theta) \otimes H^n(X, 8\Theta) \rightarrow H^n(X, 16\Theta)$$

is surjective. Considering the zero character space and $\pm$ eigenspaces, we will prove (2). \qed

Finally, let us denote by Heis the Heisenberg group associated to the line bundle $\mathcal{E}(2\Theta)$ (see [12] or [2, page 280]), that is, a central extension of $X[2]$ by $\mathbb{C}^*$:

$$0 \rightarrow \mathbb{C}^* \rightarrow \text{Heis} \rightarrow X[2] \rightarrow 0.$$  

(4.8)

We recall that Heis acts linearly on $H^n(X, 2\Theta)$, its unique (up to conjugation) representation of level 1.

### 5. The Prym–Wirtinger duality

Consider $\eta, \zeta \in JC[2]$ such that $\langle \eta, \zeta \rangle = 1$. We recall the definition of the line bundle $\mathcal{L}^\zeta_\eta = \mathcal{E}_\eta(\Xi_\zeta + T^*_\eta \Xi_\zeta) = \mathcal{E}_\eta(2\Xi_\zeta \otimes \eta)$. In the next proposition we show that exchanging the roles of $\eta$ and $\zeta$ will establish a duality at the level of global sections...
of $\mathcal{L}_\eta$. We may view this duality as an analogue for Prym varieties of the well-known Wirtinger duality (put $\eta = \zeta = 0$) for Jacobians (see [14, page 335]). This Prym–Wirtinger duality is at the heart of the proof of Theorem 1.1 (Section 7), as the Prym–Spin pairing ‘restricts’ to the Prym–Wirtinger duality for suitably chosen (products of) Prym varieties in $\mathcal{M}(\text{Spin}_n)$.

**Proposition 5.1.** We have the following canonical isomorphisms for any $\eta, \zeta \in JC[2]$ such that $\langle \eta, \zeta \rangle = 1$.

\[
\begin{align*}
H^0(\mathcal{L}_\eta^\vee, \mathcal{L}_0^\vee) & \cong H^0(P_\eta^0, \mathcal{L}_0^\vee), \quad H^0(P_\eta^0, \mathcal{L}_0^\vee) \cong H^0(P_\eta^0, \mathcal{L}_0^\vee), \quad (5.1) \\
H^0(P_\eta^0, \mathcal{L}_0^\vee) & \cong H^0(P_\eta^0, \mathcal{L}_0^\vee), \quad H^0(P_\eta^0, \mathcal{L}_0^\vee) \cong H^0(P_\eta^0, \mathcal{L}_0^\vee). \quad (5.2)
\end{align*}
\]

**Proof.** We will show that the duality between the above vector spaces is given by a reduced Cartier divisor, whose set-theoretical support equals

\[
\Delta_{\eta, \zeta}^{W} := \{(\xi, L) \in (P_\eta^0 \cup P_\eta^0) \times (P_\eta^0 \cup P_\eta^0) \mid h^0(C, \pi_\eta^0 \xi \otimes \pi_\eta^0 \xi, L) > 0\}. \quad (5.3)
\]

Indeed, as shown in Example 2.2, the family $\mathcal{L}$ of rank 4 vector bundles $\pi_\eta^0 \xi \otimes \pi_\eta^0 \xi, L$ parametrized by $(P_\eta^0 \cup P_\eta^0) \times (P_\eta^0 \cup P_\eta^0)$ is equipped with an $\omega$-valued quadratic form. Hence, by Subsection 2.3, we can consider its associated Pfaffian divisor $\Delta_{\eta, \zeta}^{W} := \text{divPF}(\mathcal{L})$. (By (2.6) its support equals the set in (5.3) and an easy computation shows that $\mathcal{L}(\Delta_{\eta, \zeta}^{W}) = \mathcal{L}_\eta^\vee \otimes \mathcal{L}_\eta^\vee$. Hence we obtain a pairing

\[
H^0(P_\eta^0 \cup P_\eta^0, \mathcal{L}_0^\vee)^{\vee} \longrightarrow H^0(P_\eta^0 \cup P_\eta^0, \mathcal{L}_0). \quad (5.4)
\]

First we will show that $\psi$ is an isomorphism. We verify that both sides of (5.4) are Heis-modules (4.8) of level 1 and that $\psi$ is Heis-equivariant. Hence, since $\psi$ is non-zero, it is an isomorphism.

To finish the proof we have to analyse how $\psi$ acts on the ±eigenspaces of the linear involutions $\sigma_\eta^*$ (respectively $\sigma_\eta^*$). Let us restrict attention to the duality on the component $P_\eta^0 \times P_\eta^0$. Consider the rational map, induced by the divisor $\Delta_{\eta, \zeta}^{W}$:

\[
\begin{align*}
\Delta \colon P_\eta^0 & \longrightarrow \mathbb{P} H^0(P_\eta^0, \mathcal{L}_0^\vee) \\
\zeta & \longmapsto \Delta(\zeta) := \Delta_{\eta, \zeta}^{W} \mid \{\zeta \in P_\eta^0\}.
\end{align*}
\]

We observe that, for all $\zeta \in P_\eta^0$, the divisor $\Delta(\zeta)$ is invariant under the (projective) involution $\sigma_\eta^*$. Consider $\kappa \in \mathcal{H}(C)$ such that $\kappa \in \mathcal{H}(P_\eta^0) \cap \mathcal{H}(P_\eta^0)$, that is, $\kappa(\eta) = \kappa(\zeta) = 1$. Then it follows from the definition of $\Delta_{\eta, \zeta}^{W}$ that $\Delta(\pi_\eta^* \kappa) = T_0^* \Xi + T_0^* \Xi$. (By (2.5) and with the notation as above, we have

\[
\sigma_\eta^*(s_\eta, s_\eta) = \epsilon(\kappa) \epsilon(\eta \kappa) s_\eta \cdot s_\eta = \kappa(\eta) s_\eta \cdot s_\eta = s_\eta \cdot s_\eta.
\]

Thus we get $\Delta(\pi_\eta^* \kappa) \in \mathbb{P} H^0(P_\eta^0, \mathcal{L}_0^\vee)$. Hence $\text{im} \Delta \subset \mathbb{P} H^0(P_\eta^0, \mathcal{L}_0^\vee)$. On the other hand, we see that the divisor $\Delta(L) := \Delta_{\eta, \zeta}^{W} \mid \{\zeta \in L\}$ for $L \in P_\eta^0$ is symmetric and, take for example $L = \emptyset$, $\Delta(L) \in \mathbb{P} H^0(P_\eta^0, \mathcal{L}_0^\vee)$. Hence the pairing (5.4) splits as follows,

\[
H^0(P_\eta^0, \mathcal{L}_0^\vee)^{\vee} \longrightarrow H^0(P_\eta^0, \mathcal{L}_0^\vee).
\]

By the same reasoning as above, one gets the other three isomorphisms as stated in the proposition. We leave the details to the reader. \hfill \square
6. The Schottky variety \( \text{SCH}_n \)

In this section we define a new variety, the Schottky variety \( \text{SCH}_n \) for \( n \geq 2 \), which is inductively constructed by taking unions of products of Prym and Jacobian varieties, as well as a morphism from \( \text{SCH}_n \) into \( \mathcal{M}(\text{Spin}_3) \). The first non-trivial case \( \text{SCH}_3 \) is just the union of all Prym varieties and the Jacobian and their image in \( \mathcal{M}(\text{Spin}_3) \) is of fundamental use in the Schottky problem (see Subsection 8.1). The next case \( \text{SCH}_4 \) is modelled on the exceptional isomorphism \( \text{Spin}_4 = \text{SL}_2 \times \text{SL}_2 \). Cases \( n = 5, 6 \) are constructed from the previous ones by adding a Jacobian factor and for \( n \geq 7 \) we proceed by induction. The higher order analogues \( \text{SCH}_n \), for \( n \geq 4 \), are introduced only for the purposes of the proof of Theorem 1.1, although we suspect that they possess some interesting geometric properties like \( \text{SCH}_n \).

6.1. \( n = 2 \)

This case has been extensively discussed in [16, Section 4.2]. Unlike in [16], we will define \( \mathcal{M}(\text{Spin}_3) \) to be the degree 0 component of \( \mathcal{M}(\mathbb{C}^*) = \text{Pic}(\mathbb{C}) \), that is, \( \mathcal{M}(\text{Spin}_3) = \mathcal{J} \). We also put \( \mathcal{M}(\text{Spin}_3) = \emptyset \). We define the Schottky variety by \( \text{SCH}_2 = \mathcal{J} \mathcal{C} \) and the map \( \phi_2 \) is the identity \( \phi_2: \mathcal{J} \mathcal{C} = \mathcal{M}(\text{Spin}_3) \). Given \( A \in \mathcal{M}(\text{Spin}_3) \), we denote the line bundle \( A(S^*) \in \mathcal{J} \mathcal{C} \) by \( N \). Then we have \( A(S^*) = N^{-1} \) and \( A(V) = N^2 \oplus N^{-2} \).

6.2. \( n = 3 \)

In this case (see [16, Section 4.3] we have the isomorphisms \( \mathcal{M}(\text{Spin}_3) = \mathcal{U}(2, \mathcal{O}) \) and \( \mathcal{M}(\text{Spin}_3) = \mathcal{U}(2, \mathcal{O}(p)) \), where we used the exceptional isomorphism \( \text{Spin}_3 \cong \text{SL}_2 \) at the level of algebraic groups. Here \( \mathcal{U}(2, L) \) denotes the moduli space of semistable rank 2 vector bundles with fixed determinant equal to \( L \in \text{Pic}(\mathbb{C}) \). To a semistable \( \text{Spin}_3 \)-bundle \( A \in \mathcal{M}(\text{Spin}_3) \) we associate two semistable vector bundles \( A(S) \) (respectively \( A(V) \)) induced by the spinor (respectively orthogonal) representation. In particular the isomorphism

\[
\mathcal{M}(\text{Spin}_3) \cong \mathcal{U}(2, \mathcal{O})
\]

is given by the map \( A \mapsto A(S) \). Then the orthogonal rank 3 bundle is given by

\[
A(V) = A(S^*). \]

We choose a point \( \bar{\rho} \in \mathcal{C}_1 \) lying over \( p \). Taking direct image gives the following maps

\[
\phi_3^{-}(\overline{\cdot}^{\rho}) : P_3^\zeta \longrightarrow \mathcal{M}(\text{Spin}_3) \quad \phi_3^{+}(\overline{\cdot}^{\rho}) : P_3^\zeta \longrightarrow \mathcal{M}(\text{Spin}_3) \\
L \longrightarrow \pi_\zeta, L = A(S) \quad L \longrightarrow \pi_\zeta, L(\bar{\rho}) = A(S). 
\]

If \( \zeta = 0 \), we define \( \phi_3^{-}(L) = L \oplus L^{-1} \) and \( \phi_3^{+}(L) \) is the unique stable rank 2 bundle, which fits into the exact sequence (see [2, Section 3])

\[
0 \longrightarrow L \oplus L^{-1} \longrightarrow \phi_3^{-}(L) \longrightarrow \mathcal{C}_p \longrightarrow 0.
\]

We define the Schottky variety and the morphism \( \phi_3^\pm : \text{SCH}_3 \longrightarrow \mathcal{M}(\text{Spin}_3) \).
6.3. \( n = 4 \)

In this case (see [16, Section 4.4]) we have the isomorphisms \( \mathcal{M}^+(\text{Spin}_4) = \mathcal{M}^-(\text{Spin}_4) \times \mathcal{M}^- (\text{Spin}_4) \) and \( \mathcal{M}^- (\text{Spin}_4) = \mathcal{M}^- (\text{Spin}_4) \times \mathcal{M}^+ (\text{Spin}_4) \), where we used the exceptional isomorphism \( \text{Spin}_4 = \text{SL}_2 \times \text{SL}_2 \). The previous isomorphism is given by sending a Spin\(_4\)-bundle \( A \) to the pair of rank 2 bundles \((A(S'^*), A(S^*))\). Consider the maps

\[
\phi^\pm_{\xi, \zeta} : P^\xi_{\zeta} \times P^\zeta_{\xi} \longrightarrow \mathcal{M}^\pm (\text{Spin}_4)
\]

where we use the previous isomorphisms. We define the Schottky variety and the morphism \( \phi^\pm_{\xi, \zeta} \) by

\[
\prod \phi^\pm_{\xi, \zeta} : \text{SCH}_4 := \prod_{\zeta \in \mathcal{C}[2]} P^\xi_{\zeta} \times P^\zeta_{\xi} \longrightarrow \mathcal{M}^\pm (\text{Spin}_4).
\]

6.4. \( n = 5 \)

We define \( \phi^\pm_{\xi, \zeta} \) to be the composite map (see Subsections 6.1, 6.2 and Lemma 3.1)

\[
P^\xi_{\zeta} \times JC \longrightarrow \mathcal{M}^\pm (\text{Spin}_4) \times \mathcal{M}^+ (\text{Spin}_2) \longrightarrow \mathcal{M}^\pm (\text{Spin}_4).
\]

For example, given \((L, N) \in P^\xi_{\zeta} \times JC\), the Spin\(_5\)-bundle \( A := \phi^\pm_{\xi, \zeta} (L, N) \) satisfies

\[
A(S) = (N \oplus N^{-1}) \oplus \pi_\zeta, L, \quad A(V) = \xi \oplus N^2 \oplus N^{-2} \oplus \pi_\zeta, L^2.
\]

(6.1)

We remark that in this case we have an isomorphism \( \mathcal{M}^+ (\text{Spin}_5) = \mathcal{M}^+ (\text{Spin}_4) \), \( A \longrightarrow A(S) \). The symplectic form on the bundle \( A(S) \) in (6.1) is the obvious one. We define the Schottky variety by

\[
\phi^\pm_{\xi, \zeta} : \text{SCH}_5 := \prod_{\zeta \in \mathcal{C}[2]} P^\xi_{\zeta} \times JC \longrightarrow \mathcal{M}^\pm (\text{Spin}_5).
\]

6.5. \( n = 6 \)

Consider the composite map (see Subsections 6.1, 6.3 and Lemma 3.1)

\[
\phi^\pm_{\xi, \zeta} : P^\xi_{\zeta} \times P^\zeta_{\xi} \times JC \longrightarrow \mathcal{M}^\pm (\text{Spin}_4) \times \mathcal{M}^+ (\text{Spin}_2) \longrightarrow \mathcal{M}^\pm (\text{Spin}_6)
\]

which associates to the triple \((L, M, N) \in P^\xi_{\zeta} \times P^\zeta_{\xi} \times JC\) the Spin\(_6\)-bundle \( A \), which verifies

\[
A(V) = N^2 \oplus N^{-2} \oplus \pi_\zeta, LM \oplus \pi_\zeta, LM^{-1},
\]

\[
A(S) = (N \oplus \pi_\zeta, M) \oplus (N^{-1} \oplus \pi_\zeta, L), \quad A(S^*) = (N \oplus \pi_\zeta, L) \oplus (N^{-1} \oplus \pi_\zeta, M).
\]

We define the Schottky variety by

\[
\phi^\pm_{\xi, \zeta} : \text{SCH}_6 := \prod_{\zeta \in \mathcal{C}[2]} P^\xi_{\zeta} \times P^\zeta_{\xi} \times JC \longrightarrow \mathcal{M}^\pm (\text{Spin}_6).
\]

6.6. The general case

Now we will define by induction the Schottky variety \( \text{SCH}_n \) for any integer \( n \). We put for \( n \geq 3 \)

\[
\text{SCH}_{n+4, \xi, \zeta} := P^\xi_{\zeta} \times P^\zeta_{\xi} \times \text{SCH}_n, \quad \text{SCH}_{n+4} := \prod_{\zeta \in \mathcal{C}[2]} \text{SCH}_{n+4, \xi, \zeta}
\]
and define on each component $SCH_{n+4,\xi}$ the morphism $\phi_{n+4,\xi}^\pm$ as the composite map (see Lemma 3.1)

$$SCH_{n+4,\xi} \xrightarrow{\phi_{n+4,\xi}^\pm \times \phi^\pm_{\xi}} \mathcal{M}^*(\text{Spin}_n) \times \mathcal{M}^\pm(\text{Spin}_n) \longrightarrow \mathcal{M}^\pm(\text{Spin}_{n+4}).$$

Note that for $n \geq 5$ the image of $SCH_n$ is contained in the semistable boundary of $\mathcal{M}^\pm(\text{Spin}_n)$.

**Remark 6.1.** Note that, if we put $n = 2$ in the general definition, we recover the Schottky variety $SCH_2$ as introduced in Subsection 6.5, but not the morphism $\phi_2^\pm$ into $\mathcal{M}^\pm(\text{Spin}_2)$. We recall that $\mathcal{M}^\pm(\text{Spin}_2) = \emptyset$.

**Remark 6.2.** A different (and more intrinsic) way of generalizing the Schottky variety $SCH_2$ is to construct natural algebraic groups $R_n$ with morphisms into $\text{Spin}_n$ and to consider the moduli spaces $\mathcal{M}(R_n)$ of semistable principal $R_n$-bundles. We recall that $SCH_2 = \mathcal{M}(\text{Pin}_2)$ [15].

### 7. Proof of Theorem 1.1

Before launching into the technical calculations, we give a brief overview of the main points of the proof.

1. We compose the map $\sum s_\eta^\pm$ of Theorem 1.1 with the pull-back under $\phi_n^\pm$ to the Schottky variety $SCH_n$ constructed in Section 6.
2. On each product $P_\eta \times SCH_{n,\xi}$ the pull-back of the divisor $\mathcal{D}_\eta^\pm$ splits (Lemmas 7.1 and 7.3), which implies a factorization of the composite map of 1 (Lemmas 7.2 and 7.4).
3. Summing over $\eta, \xi \in JC[2]$, we observe that this factorization is essentially the composite of
   - (i) multiplication maps, discussed in Section 4;
   - (ii) Prym–Wirtinger duality (Proposition 5.1).
4. Since the Schottky varieties $SCH_n$ are constructed inductively (Subsections 6.3 and 6.4), we first will prove the cases $n \leq 6$ separately and then proceed by induction.

We will concentrate on the cases $n = 3, 4$ because the proofs for $n = 5, 6$ as well as the induction go along the same lines. For completeness we include the following.

#### 7.1. $n = 2$

For the details see [16, Section 4.2]. In this case $\mathcal{M}^\pm(\text{Spin}_2) = JC$ and $\Theta(C^2) = \mathcal{O}_{JC}(8\Theta)$. The morphism (1.2) decomposes as follows. By the Prym–Wirtinger duality (5.1) with $\zeta = 0$, we have

$$\sum_{\eta \in JC[2]} H^0(P_\eta^\text{ev}, 2\Xi_\eta)^\vee \oplus H^0(P_\eta^\text{odd}, 2\Xi_\eta)^\vee = \sum_{\eta \in JC[2]} H^0(JC, 2\Theta \otimes \eta) \oplus H^0(JC, 2\Theta \otimes \theta),$$

which by (4.5) equals $H^0(JC, 8\Theta)$. 

7.2. $n = 3$

First we shall consider the duality on $\mathcal{H}(\text{Spin}_g)$. The following lemma will tell us how $\mathcal{D}_q^\perp$ splits over $\text{SCH}_a$.

**Lemma 7.1.** Given $M \in \text{Pic}(\tilde{\mathcal{C}})$. Then the orthogonal bundle $\text{End}_q(\pi_\zeta, M)$ is an orthogonal direct sum

$$\text{End}_q(\pi_\zeta, M) = \zeta \oplus \pi_\zeta(M^2 \otimes \pi_\zeta^*(\text{Nm} M)^{-1}).$$

In particular, for $L \in P^\zeta_q$,

$$\text{End}_q(\pi_\zeta, L) = \zeta \oplus \pi_\zeta(L^2). \quad \text{End}_q(\pi_\zeta, (L(\tilde{\mu} - \sigma \tilde{\mu}))) = \zeta \oplus \pi_\zeta(L^3(\tilde{\mu} - \sigma \tilde{\mu})) \quad \text{(7.1)}$$

**Proof.** Since $\pi^*_\zeta \zeta = \mathcal{O}_c$, we have an isomorphism

$$\zeta \otimes \pi_\zeta, M \sim \pi_\zeta, M,$$

which gives rise to a homomorphism $\zeta \longrightarrow \text{End}_q(\pi_\zeta, M)$. Since both $\zeta$ and $\text{End}_q(\pi_\zeta, M)$ have degree 0 and since $\text{End}_q(\pi_\zeta, M)$ is poly-stable, it follows that $\zeta$ is a subbundle of $\text{End}_q(\pi_\zeta, M)$. It remains to determine its supplement bundle. There is a natural homomorphism of $\text{Sym}^2(\pi_\zeta, M)$ into $\pi_\zeta, M^2$. Using the isomorphism $\text{End}_q(\pi_\zeta, M) \cong \text{Sym}^2(\pi_\zeta, M) \otimes (\text{Nm} M)^{-1} \otimes \zeta$, we get a homomorphism of $\text{End}_q(\pi_\zeta, M)$ into $\pi_\zeta, M^2 \otimes (\text{Nm} M)^{-1}$. A pointwise check shows that this is a surjective homomorphism and that it is supplementary to $\zeta$. \(\square\)

By the first equality in (7.1), we have for all $L \in P^\zeta_q$, $\zeta \in P^\mu_q$,

$$h^0(C, \text{End}_q(\pi_\zeta, L) \otimes \pi^*_\zeta, \zeta) = h^0(\tilde{\mathcal{C}}, \zeta \otimes \pi^*_\zeta, \zeta) + h^0(C, \pi_\zeta, L^2 \otimes \pi^*_\zeta, \zeta). \quad \text{(7.2)}$$

Hence we claim that the pull-back of the divisor $\mathcal{D}_q^\perp (3.3)$ by the morphism

$$(\text{id} \times \phi^*_\zeta, P^\mu_q \times P^\zeta_q \longrightarrow P^\mu_q \times \mathcal{H}(\text{Spin}_g)$$

splits into two divisors

$$(\text{id} \times \phi^*_\zeta, P^\mu_q \times P^\zeta_q \longrightarrow P^\mu_q \times \mathcal{H}(\text{Spin}_g))$$

where $p_1$ is the projection onto the first factor $P^\mu_q$, $\text{id} \times \phi^*_\zeta, P^\mu_q \longrightarrow P^\mu_q$ is the duplication map $L \longrightarrow L^2$, and $\Delta_{\zeta, \zeta}^\text{w, ir}$ is the divisor defined in (5.3). Indeed, by (7.2) the decomposition (7.3) follows set-theoretically and since $(\text{id} \times \phi^*_\zeta, P^\mu_q \times \mathcal{D}_q^\perp) \in [3 \Xi_q \otimes 8 \Xi_q]$, the equality (7.3) also holds scheme-theoretically. The next lemma is an immediate consequence of this decomposition.

**Lemma 7.2.** For any $\eta, \zeta$ satisfying $\langle \eta, \zeta \rangle = 1$, the linear map $s^*_\zeta$ composed with the pull-back induced by $\phi^*_\zeta, \zeta$,

$$H^q(P^\mu_q, 3 \Xi_q)^\vee \longrightarrow H^0(\mathcal{H}, \Theta^{\mathcal{C}}) \longrightarrow H^0(P^\mu_q, 8 \Xi_q),$$

factorizes as

$$H^q(P^\mu_q, 3 \Xi_q)^\vee \longrightarrow H^0(P^\mu_q, \mathcal{D}_q^\perp) \longrightarrow H^0(P^\mu_q, 8 \Xi_q),$$

where the first map is the dual of the multiplication map $D \longrightarrow D + T^\perp \Xi_q$.\[524\]
We are now in a position to prove (1.1) for \( m = 1 \). The main idea of the proof is to show that the composite map

\[
\sum_{\eta} H^0(P^\eta_\eta, \mathcal{F}^\chi_\eta) \rightarrow H^0(\mathcal{M}^\chi_{\text{Spin}}, \mathcal{F}(\mathbb{C}^\eta)) \rightarrow \sum_{\xi} H^0(P^\xi_\eta, 8\Xi_\eta)_+ (7.4)
\]

is injective, which immediately implies that \( \sum s^\eta_\eta \) is an isomorphism, since the first two spaces have the same dimension \([16, \text{Theorem 3.1}]\).

By Lemma 7.2, the linear map (7.4) factorizes as

\[
\sum_{\eta \in JC[2]} H^0(P^\eta_\eta, \mathcal{F}^\chi_\eta) \rightarrow \sum_{\eta \in JC[2]} H^0(P^\eta_\eta, \mathcal{L}_\eta^\nu) \rightarrow \sum_{\xi \in JC[2]} H^0(P^\xi_\eta, 8\Xi_\eta)_+.
\]

The arrows are as follows: (1) is the dual of the multiplication map in proposition 4.2(1), which is injective; (2) is the Prym–Wirtinger duality (5.1), which is an isomorphism, and (3) is the isomorphism induced by the duplication map (4.5).

Finally we observe that we can invert the indices of summation in (2), since both sets of indices are in one-to-one correspondence with the set of isotropic (with respect to the Weil form) Klein subgroups of \( JC[2] \). Since the three maps (1),(2),(3) are injective, their composite map (7.4) is also injective and we are done.

Let us briefly indicate how to adapt the previous proof to the moduli \( \mathcal{M}^\chi_{\text{Spin}} \). First, using the second equality in (7.1), we easily see that the analogue of (7.3) is

\[(\text{id} \times \phi^\ast_{\alpha, \chi}(\mathcal{D}_\chi)) = p^\ast_{\gamma}(T^\ast_\gamma \Xi_\gamma) + (\text{id} \times (T^\ast_\gamma \circ [2]))^\ast (\Lambda^\nu_{\alpha, \chi}),\]

where \( T^\ast_\gamma : P^\ast_\gamma \rightarrow P^\ast_\gamma \) denotes translation by \( \epsilon_\gamma(\delta^\gamma - \sigma^\gamma) + \epsilon_\gamma \). Let us choose a square root \( \delta \in P^\ast_\gamma \) of \( \epsilon_\gamma(\delta^\gamma - 2\sigma^\gamma) + \epsilon_\gamma \). Then we have the equality \( T^\ast_\gamma \mathcal{L}_\gamma = T^\ast_\gamma \mathcal{L}_\gamma \) among line bundles over \( P^\ast_\gamma \). We also recall the equality \( [2] \circ T = T \circ [2] \), where \( e \in P^\ast_\gamma \) is a square root of \( \delta \). Using this notation, one easily verifies that the composite map \( \phi^\ast_{\alpha, \chi} \circ \eta^\ast_\gamma \) factorizes as follows (analogue of Lemma 7.2):

\[
H^0(P^\ast_\eta, \mathcal{F}^\chi_\eta) \rightarrow H^0(P^\ast_\eta, \mathcal{L}_\eta^\nu) \rightarrow H^0(P^\ast_\xi, \mathcal{L}_\xi^\nu) (7.5)
\]

We note that the linear isomorphism \( (T^\ast_\gamma \circ T)^\ast \) depends on the choice of the square root \( \delta, \), which implies the indeterminacy in the sign \( \pm \) of the eigenspaces. This sign is irrelevant for the rest of the proof. The factorization (7.5) allows us to conclude as above.

7.3. \( n = 4 \)

As in the previous subsection, we first consider the duality on \( \mathcal{M}^\chi_{\text{Spin}} \). The following lemma is the analogue of Lemma 7.1.
Lemma 7.3. Given a pair \((L, M) \in P^\xi_\ell \times P^\xi_\ell\), we consider their associated Spin\(_4\)-bundles \(A^\pm := \phi^\pm_\xi(L, M)\). Then we have

\[ A^+(V) = \pi_\xi^*(LM) \oplus \pi^*(LM^{-1}), \quad A^-(V) = \pi_\xi^*(LM(\beta - \sigma \bar{\beta})) \oplus \pi^*(LM^{-1}). \]

Proof. We know that the orthogonal bundle \(A^\pm(V)\) is the tensor product \(A^\pm(S^+)^\vee \otimes A^\pm(S^-)^\vee\). Let us do the computations for \(A^+(V)\):

\[ A^+(V) = A^+(S^+) \otimes A^+(S^-) = \pi_\xi^* L \otimes \pi^* M = \pi_\xi^* (L \otimes \pi^* \pi^*_\xi M), \]

where we used the fact that \(A^+(S^-)\) is self-dual and the projection formula for the map \(\pi\). Now by Example 2.1, \(\pi^*_\xi \pi_\xi M\) is a semistable orthogonal bundle over \(C\), hence this bundle splits \(\pi^*_\xi M = M \oplus M^{-1}\). The computations for \(A^-(V)\) are similar. \(\square\)

We immediately get for all \((L, M) \in P^\xi_\ell \times P^\xi_\ell\), all \(\xi \in P^\text{ev}_\tau \cup P^\text{odd}_\tau\):

\[ h^0(\xi, A(V) \otimes \pi_\xi, \xi) = h^0(\xi, \pi_\xi^*(LM) \otimes \pi_\eta^* \xi) + h^0(\xi, \pi_\xi^*(LM^{-1}) \otimes \pi_\eta^* \xi). \]

Hence the pull-back of the divisor \(\mathcal{D}_\tau^+ (3.3)\) by the morphism

\[ (\text{id} \times \phi^+_{\xi, \eta}) : (P^\text{ev}_\eta \cup P^\text{odd}_\eta) \times (P^\xi_\ell \times P^\xi_\ell) \longrightarrow (P^\text{ev}_\eta \cup P^\text{odd}_\eta) \times \mathcal{M}^+(\text{Spin}_4) \]

splits into two divisors

\[ (\text{id} \times \phi^+_{\xi, \eta})^* (\mathcal{D}_\tau^+) = (\text{id} \times m)^* [p^\text{ev}_{12} \Lambda^{\text{wrt}}_{\eta, \xi} + p^\text{odd}_{13} \Lambda^{\text{wrt}}_{\eta, \xi}], \quad (7.6) \]

where \(m\) is the isogeny \(P^\xi_\ell \times P^\xi_\ell \longrightarrow P^\text{ev}_\tau \times P^\text{odd}_\tau\) defined in (4.1) and \(p_{ij}\) is the projection of \((P^\text{ev}_\eta \cup P^\text{odd}_\eta) \times P^\xi_\ell \times P^\xi_\ell\) onto the \(i\)th and \(j\)th factors. The decomposition (7.6) leads to the following factorization.

Lemma 7.4. For any \(\eta, \xi\) satisfying \(\langle \eta, \xi \rangle = 1\), the linear map \(s^\eta_\xi\) composed with the pull-back induced by \(\phi^+_{\xi, \eta}\)

\[ H^0(P^\text{ev}_\eta, 4\Xi_\eta)^\vee \oplus H^0(P^\text{odd}_\eta, 4\Xi_\eta)^\vee \longrightarrow H^9(\mathcal{M}^+(\text{Spin}_4), \Theta(\mathbb{C}^4)) \]

factorizes as follows:

\[ H^9(P^\text{ev}_\eta \Gamma M, 4\Xi_\eta, 4\Xi_\ell) \oplus H^9(P^\text{odd}_\eta, 4\Xi_\eta, 4\Xi_\ell) \]

\[ \longrightarrow [H^9(P^\text{ev}_\eta, \mathcal{L}^\eta, \xi) \otimes H^9(P^\text{ev}_\eta, \mathcal{L}^\eta, \xi)^\vee] \oplus [H^9(P^\text{odd}_\eta, \mathcal{L}^\eta, \xi) \otimes H^9(P^\text{odd}_\eta, \mathcal{L}^\eta, \xi)^\vee] \]

\[ \longrightarrow [H^9(P^\text{ev}_\eta, \mathcal{L}^\eta, \xi) \otimes H^9(P^\text{ev}_\eta, \mathcal{L}^\eta, \xi)^\vee] \oplus [H^9(P^\text{odd}_\eta, \mathcal{L}^\eta, \xi) \otimes H^9(P^\text{odd}_\eta, \mathcal{L}^\eta, \xi)^\vee] \]

\[ \longrightarrow m^* H^9(P^\xi_\ell \times P^\xi_\ell, 4\Xi_\eta \otimes 4\Xi_\ell) \]

where the first map is the dual of the multiplication map.
As in the $n = 3$ case, it will be enough to show that the composite map

$$
\sum_{\eta} H^0(P^+_{\eta}, 4\Xi_\eta)^\vee \oplus H^0(P^{\text{odd}}_{\eta}, 4\Xi_\eta)^\vee \xrightarrow{\phi_{\xi}^*} H^0(\mathcal{M}^\ast(\text{Spin}_4), \Theta(C^\xi))
$$

is injective. By Lemma 7.4 this linear map (7.7) factorizes as follows:

$$
\sum_{\eta \in \tilde{H}(\mathcal{T}^2)} H^0(P^+_{\eta}, 4\Xi_\eta)^\vee \oplus H^0(P^{\text{odd}}_{\eta}, 4\Xi_\eta)^\vee
$$

$$(1) \xrightarrow{(1)} \sum_{\eta \in \tilde{H}(\mathcal{T}^2)} \sum_{\xi \in \tilde{H}(\mathcal{T}^2)} [H^0(P^+_{\eta}, \mathcal{L}^\xi)\vee \otimes H^0(P^{\text{odd}}_{\eta}, \mathcal{L}^\xi)\vee] \oplus [H^0(P^{\text{odd}}_{\eta}, \mathcal{L}^\xi)\vee \otimes H^0(P^+_{\eta}, \mathcal{L}^\xi)\vee]
$$

$$(2) \xrightarrow{(2)} \sum_{\xi \in \tilde{H}(\mathcal{T}^2)} \sum_{\eta \in \tilde{H}(\mathcal{T}^2)} [H^0(P^+_{\eta}, \mathcal{L}^\xi)\vee \otimes H^0(P^{\text{odd}}_{\eta}, \mathcal{L}^\xi)\vee] \oplus [H^0(P^{\text{odd}}_{\eta}, \mathcal{L}^\xi)\vee \otimes H^0(P^+_{\eta}, \mathcal{L}^\xi)\vee]
$$

$$(3) \xrightarrow{(3)} \sum_{\xi \in \tilde{H}(\mathcal{T}^2)} H^0(P^+_{\eta}\times P^+_{\eta}, 4\Xi_\eta \boxtimes 4\Xi_\eta)^\vee.
$$

Map (1) is the dual of the multiplication map in Proposition 4.2(2), hence is injective. Map (2) is the Prym–Wirtinger duality (5.1), and map (3) is the isomorphism (4.3). Hence the composite map is injective.

In order to avoid repetition, we will just indicate the changes to be done to adapt the previous proof to $\mathcal{M}^\ast(\text{Spin}_4)$. The analogue of (7.6) is

$$
(\text{id} \times \phi_{\xi, \eta}^\ast)(\mathcal{D}_\eta) = (\text{id} \times (T_\rho \times \text{id}) \circ m)^\ast [p_{12}^\ast \Delta_{\eta, \xi} + p_{13}^\ast \Delta_{\eta, \xi}^\text{Wirt}],
$$

where the right-hand side divisor is taken in $(P^+_{\eta}\cup P^{\text{odd}}_{\eta}) \times P^+_{\eta}\times P^+_{\eta}$. This implies that the composite map $\phi_{\xi, \eta}^\ast \circ s_\eta$ factorizes as follows (analogue of Lemma 7.4):

$$
H^0(P^+_{\eta}, 4\Xi_\eta)^\vee \oplus H^0(P^{\text{odd}}_{\eta}, 4\Xi_\eta)^\vee
$$

$$
\xrightarrow{(1)} \left[H^0(P^+_{\eta}, \mathcal{L}^\xi)\vee \otimes H^0(P^{\text{odd}}_{\eta}, \mathcal{L}^\xi)\vee \right] \oplus \left[H^0(P^{\text{odd}}_{\eta}, \mathcal{L}^\xi)\vee \otimes H^0(P^+_{\eta}, \mathcal{L}^\xi)\vee \right]
$$

$$
\xrightarrow{(2)} \left[H^0(P^+_{\eta}, \mathcal{L}^\xi)\vee \otimes H^0(P^{\text{odd}}_{\eta}, \mathcal{L}^\xi)\vee \right] \oplus \left[H^0(P^{\text{odd}}_{\eta}, \mathcal{L}^\xi)\vee \otimes H^0(P^+_{\eta}, \mathcal{L}^\xi)\vee \right]
$$

$$
\xrightarrow{(3)} H^0(P^+_{\eta}\times P^+_{\eta}, 4\Xi_\eta \boxtimes 4\Xi_\eta)^\vee \xrightarrow{(m^\ast \times T_\rho)^\ast \circ \text{id}} H^0(P^+_{\eta}\times P^+_{\eta}, T_\rho^* 4\Xi_\eta \boxtimes T_\rho^* 4\Xi_\eta)^\vee,
$$

where we used the equality $(T_\rho \times \text{id}) \circ m = m \circ (T_\rho \times T_\rho)$.

Now we conclude as above.

**7.4. $n = 5$**

Since the method of the proof is the same as for $n = 3$ and $n = 4$, we will just indicate the main steps. Again we start with $\mathcal{M}^\ast(\text{Spin}_4)$. As a consequence of (6.1), we see that the pull-back of the divisor $\mathcal{D}_\eta$ by the morphism $\text{id} \times \phi_{\xi, \eta}^\ast$ splits as follows (notation as above):

$$
(\text{id} \times \phi_{\xi, \eta}^\ast)(\mathcal{D}_\eta) = p_{12}^\ast(T_\rho^* 4\Xi_\eta) + p_{13}^\ast((\text{id} \times [2]^\ast)(\Delta_{\eta, \xi}^\text{Wirt})) + p_{14}^\ast((\text{id} \times [2]^\ast)(\Delta_{\eta, \xi}^\text{Wirt})).
$$
This decomposition implies that the composite map $\phi_n^* \circ s_n^+$ factorizes as follows:

$$
\begin{align*}
H^0(P^\text{ev}, 5\Xi_\eta) \xrightarrow{n^\text{th} \text{factor}} & H^0(P^\text{ev}, 2\Xi_\eta)^{\vee} \otimes H^0(P_\eta^\text{ev}, \mathcal{L}_\eta^\text{ev})^{\vee} \\
\xrightarrow{\text{(5.1)}} & H^0(JC, 2\Theta \otimes \eta), \otimes H^0(P_\eta^\text{ev}, \mathcal{L}_\eta^\text{ev})^{\vee} \\
\xrightarrow{\text{(6.5)}} & H^0(JC, 8\Theta), \otimes H^0(P_\eta^\text{ev}, 8\Xi_\eta).)
\end{align*}
$$

As in the previous subsections, the composite map $\phi_n^* \circ (\sum s_n^+)$ factorizes:

$$
\begin{align*}
\sum_{\eta \in JC[2]} H^0(P_\eta^\text{ev}, 5\Xi_\eta)^{\vee} \xrightarrow{(1)} & \sum_{\eta \in JC[2]} H^0(P_\eta^\text{ev}, 2\Xi_\eta)^{\vee} \otimes H^0(P_\eta^\text{ev}, 3\Xi_\eta)^{\vee} \\
\xrightarrow{(2)} & \sum_{\eta \in JC[2]} \sum_{\eta \in JC[2]} H^0(P_\eta^\text{ev}, 2\Xi_\eta)^{\vee} \otimes H^0(P_\eta^\text{ev}, \mathcal{L}_\eta^\text{ev})^{\vee} \\
\xrightarrow{(3)} & \sum_{\eta \in JC[2]} H^0(JC, 2\Theta \otimes \eta), \otimes H^0(P_\eta^\text{ev}, \mathcal{L}_\eta^\text{ev})^{\vee} \\
\xrightarrow{(4)} & \sum_{\eta \in JC[2]} H^0(JC, 8\Theta), \otimes H^0(P_\eta^\text{ev}, 8\Xi_\eta).
\end{align*}
$$

Map (1) is the dual of the (even part) multiplication map in Proposition 4.1(1) ($l = 2, m = 3$). Map (2) is the dual of the multiplication map in Proposition 4.2(1) tensored with $H^0(P_\eta^\text{ev}, 2\Xi_\eta)^{\vee}$. Map (3) is Prym–Wirtinger duality (5.1) (take $\xi = 0$ on the first factor), and map (4) is an injection, by (4.5). Since all four linear maps are injective, we are done.

The proof of the duality for $\mathcal{M}^-(\text{Spin}_n)$ does not present any difficulty and we leave it to the reader.

7.5. $n = 6$

The duality for $\mathcal{M}^+(\text{Spin}_n)$ can be proved by induction (see next subsection) since we know that it holds for $\mathcal{M}^-(\text{Spin}_n)$. Since $\mathcal{M}^+(\text{Spin}_n) = \emptyset$, we have to deal with this case separately. Again the proof is similar to the previous ones and we will omit it.

7.6. The general case

In this subsection we shall prove Theorem 1.1 by induction on $n$. We will also denote by $\Theta(V)$ the pull-back by $\phi_n^+$ of the determinant line bundle $\Theta(V) = \Theta(C^n)$ to the Schottky variety $\text{SCH}_n$. Our induction hypothesis $\mathcal{H}_n$ will be as follows.

$\mathcal{H}_n$ for $n$ odd:

$$
\phi_2^+ \circ \left( \sum s_2^+ \right) : \sum_{\eta} H^0(P_\eta^\text{ev}, n\Xi_\eta)^{\vee} \to H^0(\text{SCH}_n, \Theta(V)) \text{ is injective.}
$$

$\mathcal{H}_n$ for $n$ even:

$$
\phi_2^+ \circ \left( \sum s_2^+ \right) : \sum_{\eta} H^0(P_\eta^\text{ev}, n\Xi_\eta)^{\vee} \oplus H^0(P_\eta^\text{odd}, n\Xi_\eta)^{\vee} \to H^0(\text{SCH}_n, \Theta(V)) \text{ is injective.}
$$
Since [16, Theorem 3.1] the left-hand side space and $H^n(\mathcal{M}(\text{Spin}_n), \Theta(V))$ have the same dimension (see Subsection 8.3 for $\mathcal{M}(\text{Spin}_n)$), the assumption $\mathcal{H}_n$ implies Theorem 1.1. We already proved $\mathcal{H}_n$ for $2 \leq n \leq 6$. Let us assume $\mathcal{H}_n$ and prove $\mathcal{H}_{n+4}$.

First we easily verify that the pull-back of the divisor $\mathcal{D}_{n+4}^\pm$ under the natural map (for example for $n$ odd) (see Lemma 3.1):

$$
\text{id} \times i: P_q^\text{ev} \times \mathcal{M}(\text{Spin}_n) \times \mathcal{M}(\text{Spin}_{n+4}) \longrightarrow P_q^\text{ev} \times \mathcal{M}(\text{Spin}_{n+4})
$$

splits into two divisors

$$
(id \times i)^* (\mathcal{D}_{n+4}^\pm) = p_{22}^* (\mathcal{D}_{n}^\pm) + p_{23}^* (\mathcal{D}_{n+4}^\pm).
$$  

(7.8)

We distinguish two cases.

**Case 1: $n$ odd.**

As a consequence of the decomposition (7.8) and the proof of the $n = 4$ case, the map $\phi_{n+4} \circ (\sum s_{q}^\text{ev})$ factorizes as follows:

$$
\sum_{q \in J(\mathcal{C})[2]} H^n(P_q^\text{ev}, (n+4) \Xi_q)^\vee \\
\overset{(1)}{\longrightarrow} \sum_{q \in J(\mathcal{C})[2]} H^n(P_q^\text{ev}, 4\Xi_q)^\vee \otimes H^n(P_q^\text{ev}, n\Xi_q)^\vee \\
\overset{(2)}{\longrightarrow} \sum_{q \in J(\mathcal{C})[2]} \sum_{p \in P_q^\text{ev}[2]} H^n(P_q^\text{ev}, \mathcal{L}_p^\text{ev})^\vee \otimes H^n(P_q^\text{ev}, \mathcal{L}_p^\text{even})^\vee \otimes H^n(P_q^\text{ev}, n\Xi_q)^\vee \\
\overset{(3)}{\longrightarrow} \sum_{\langle x, x \rangle \in \mathcal{C}[2]} \sum_{q \in P_x^\text{even}[2]} H^n(P_x^\text{even}, \mathcal{L}_x^\text{odd}) \otimes H^n(P_x^\text{even}, \mathcal{L}_x^\text{even}) \otimes H^n(P_q^\text{ev}, n\Xi_q)^\vee \\
\overset{(4)}{\longrightarrow} \sum_{\langle x, x \rangle \in \mathcal{C}[2]} \left[ \sum_{q \in P_x^\text{even}[2]} H^n(P_x^\text{even}, \mathcal{L}_x^\text{even}) \otimes H^n(P_x^\text{even}, \mathcal{L}_x^\text{odd}) \right] \otimes \left[ \sum_{q \in J(\mathcal{C})[2]} H^n(P_q^\text{ev}, n\Xi_q)^\vee \right] \\
\overset{(5)}{\longrightarrow} \sum_{\langle x, x \rangle \in \mathcal{C}[2]} H^n(P_x^\text{even}, 4\Xi_x \otimes 4\Xi_x) \otimes H^n(SCH_{n+4}, \Theta(V)) = H^n(SCH_{n+4}, \Theta(V)).
$$

The maps are as follows: (1) is the dual of the multiplication map in Proposition 4.1(2) ($l = 2, m = n$), if $n = 2$, we use Proposition 4.1(1) ($l = 2, m = 4$); (2) is the dual of the (even part) multiplication map in Proposition 4.2(2); (3) is the Prym–Wirtinger duality (5.1) tensored with $H^n(P_q^\text{ev}, n\Xi_q)^\vee$; (4) is the inclusion $H^n(P_q^\text{ev}, n\Xi_q)^\vee \longrightarrow \sum_{q} H^n(P_q^\text{ev}, n\Xi_q)^\vee$; (5) is an injection coming from the direct sum decomposition (4.3) tensored with the injective map of $\mathcal{H}_n$. Finally the last equality follows from the definition of $SCH_{n+4}$. Since all linear maps are injective, the composite map is injective and we have proved $\mathcal{H}_{n+4}$.

**Case 2: $n$ even.**

Since this case is similar to the odd case, we just indicate the minor changes to be done on the sequence of maps (1)–(5). We take into account the additional factors $\sum_{q} H^n(P_q^\text{odd}, (n+4) \Xi_q)^\vee$, for which we can write down the maps (1)–(4) with $P_q^\text{ev}$ replaced by $P_q^\text{odd}$ and $H^n(P_q^\text{even}, \mathcal{L}_x^\text{even})$ by $H^n(P_x^\text{odd}, \mathcal{L}_x^\text{even})$. (see (5.1)). Thus adding the two copies of map (5) (written for $P_q^\text{ev}$ and $P_q^\text{odd}$), we observe that we still have an injection coming from (4.3).
8. Final remarks

8.1.

Let us briefly indicate why we used the name ‘Schottky variety’ for the products of Prym varieties. Indeed, for $n = 3$, the image of $\phi_3$ consists of a union of Kummer varieties of Pryms (respectively of the Jacobian) which intersect along some 4-torsion points (respectively 2-torsion points). We refer to [5, 8, 19] for a proof of these intersection properties, which may also be deduced from Lemma 7.1. The coordinates of the intersection points may be interpreted as the famous Schottky–Jung identities among theta-constants. Let us call the image $\phi_3(SCH_n) \subset \mathfrak{J}(\mathfrak{U}_n(2, \ell))$ the Schottky configuration, which we embed in projective space $\mathbb{P}^{2^g-1} = |\mathscr{L}|^\vee$, where $\mathscr{L}$ is the ample generator of $\text{Pic}(\mathfrak{J}(\mathfrak{U}_n(2, \ell)))$. In particular, we have $\Theta(\mathcal{C}^3) = \mathscr{L}^4$. Then we can deduce the following from the injectivity of (7.4).

**Corollary 8.1.** If a quartic in $\mathbb{P}^{2^g-1}$ vanishes on the Schottky configuration, then it vanishes on the whole of $\mathfrak{J}(\mathfrak{U}_n(2, \ell))$.

This corollary was proved in [8, Corollary 2]. Moreover we have been able to show that $\mathfrak{J}(\mathfrak{U}_n(2, \ell)) \subset \mathbb{P}^{2^g-1}$ is defined by quartic equations. One can thus recover $\mathfrak{J}(\mathfrak{U}_n(2, \ell))$ from the Schottky configuration purely geometrically. One might speculate whether, given the Schottky configuration of abelian varieties, one might reconstruct the curve $C$. This is the ‘small Schottky’ conjecture, see [6].

8.2.

As observed in [16, Remark 1.2] and [11, Section 7.10], the reduced divisors $D^{(n)}$, with $\kappa \in \mathfrak{J}(C)$, whose support is given by

$$\text{supp } D^{(n)} = \{ A \in \mathcal{M}^+(\text{Spin}_n) | h^n(C, A(V) \otimes \kappa) > 0 \},$$

are elements of the linear system $\mathbb{P}H^n(\mathcal{M}^+(\text{Spin}_n), \mathcal{P})$. Here $\mathcal{P}$ is a Pfaffian square root of $\Theta(V)$, that is, $\mathcal{P}^2 = \Theta(V)$. To avoid existence problems, we work over the moduli stack. Then we shall prove the following.

**Proposition 8.2.** A basis of the linear system $|\mathcal{P}|$ is given by the divisors $D^{(n)}_\kappa$ with $\kappa \in \mathfrak{J}(C)$, if $n$ is even, and $\kappa \in \mathfrak{J}(C)$ with $\varepsilon(\kappa) = 1$, if $n$ is odd.

**Proof.** It is enough to prove linear independence, which is done by induction on $n$. For the first cases we refer to [2, Proposition A.8 (n = 2), Theorem 1.2 (n = 3), Proposition A.5 (n = 4)]. The statement for $n = 5$ will follow by pulling back $D^{(5)}_\kappa \subset \mathcal{M}^+(\text{Spin}_n)$ under the map

$$\mathcal{M}^+(\text{Spin}_3) \times \mathcal{M}^+(\text{Spin}_2) \longrightarrow \mathcal{M}^+(\text{Spin}_n).$$

We observe that $i^*(D^{(5)}_\kappa) = p_1^*(D^{(3)}_\kappa) + p_2^*(D^{(2)}_\kappa)$. Since the $D^{(n)}_\kappa$ form a basis, we can conclude. Let us assume that the $D^{(n)}_\kappa$ are independent. Then the equality $i^*(D^{(n+4)}_\kappa) = p_1^*(D^{(n)}_\kappa) + p_2^*(D^{(n)}_\kappa)$ implies that the $D^{(n+4)}_\kappa$ are also independent. \qed
8.3.

The ‘twisted’ Verlinde formula given in [16, Conjecture 1.1], which computes the dimension of $H^0(\mathcal{M}^<(\text{Spin}_n), \Theta(V))$, can be deduced from a forthcoming work by Y. Laszlo and C. Sorger. Following the techniques of [11], the authors show that the ‘twisted’ Verlinde space can be identified to a conformal block, whose dimension (worked out by C. Woodward) is given by Oxbury’s formula.

8.4.

We can prove a refinement of Theorem 1.1 by considering the linear action of the Heisenberg group Heis on $H := H^0(\mathcal{M}^<(\text{Spin}_n), \Theta(V))$. By induction, we check that $H$ is a Heis-module of level 4 for the $JC[2]$-action described in (3.2). Hence the linear action of Heis factors through $JC[2]$ and we can consider its character space decomposition $H = \sum_{\eta \in JC[2]} H_{\eta}$. One proves that the image of $s_\eta^e$ is contained in $H_{\eta}$. Since $\sum s_\eta$ is surjective, we get the equality $\text{im}(s_\eta) = H_{\eta}$. Thus we have isomorphisms, for example, for $n$ odd, all $\eta \in JC[2]$,

$$s_\eta^e : H^0(P^1_\eta, (2m + 1) \Xi_\eta^{\mathbb{C}}) \times H^0(\mathcal{M}^<(\text{Spin}_2^{2m+1}), \Theta(C^{2m+1}))_{\eta}. \rightarrow H^0(\mathcal{M}^<(\text{Spin}_2^{2m+1}), \Theta(C^{2m+1}))_{\eta}.

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