HOMOLOGICAL DIMENSION OF ORE-EXTENSIONS

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Let $S$ be a ring with unit element and let $R = S[x, d]$ be the Ore-extension of $S$ with respect to a derivation $d$ of $S$. Our object in this paper is to show that $l. \text{gl. dim } R = 1 + l. \text{gl. dim } S$, if $S$ is a commutative Noetherian ring and $d$ is suitably restricted.

It was shown in [3] that $l. \text{gl. dim } R \leq 1 + l. \text{gl. dim } S$. While equality does not hold in general, we show that it does under suitable conditions (Theorem 2, § 5).

This is achieved in three steps. The first is to show that for any ring $S$, any $R$-module $M$ and an $S$-projective resolution for $M$, there exists an $R$-projective resolution of $M$ which "lifts" the given resolution (Theorem 1, § 3). The next step is to use this resolution to prove Theorem 2 in the special case in which $S$ is a local ring (Proposition 1, § 4). The final step consists in deducing Theorem 2 by the method of localisation.

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2. Preliminaries on Ore-extensions. Let $S$ be a ring with unit element (denoted by 1), which is not necessarily commutative, and let $d$ be a derivation of $S$ into itself. Let $S[x, d]$ denote the Ore-extension of $S$ with respect to $d$ (see [5]). We recall that $R = S[x, d]$ is the ring generated by an indeterminate $x$ over $S$ with the relations $xs - sx = ds$ for every $s \in S$. We identify $S$ with a subring of $R$. We collect here some properties of $R$ which will be used in the later sections.

(2.1) For any ring $S'$, a ring homomorphism $\varphi: S \to S'$ and an element $\alpha \in S'$, with the property $\alpha \varphi(s) - \varphi(s)\alpha = \varphi(ds)$, there exists a unique ring homomorphism $\overline{\varphi}: R \to S'$ such that $\overline{\varphi}(x) = \alpha$ and $\overline{\varphi}|S = \varphi$. (In fact $R$ can be characterised by this property).

The proof is straightforward.

(2.2) Let $S_1, S_2$ be rings with derivations $d_1, d_2$ respectively and let $\varphi: S_1 \to S_2$ be a ring homomorphism such that $d_2 \circ \varphi = \varphi \circ d_1$. Then there exists a ring homomorphism $\overline{\varphi}: R_1 \to R_2$ such that $\overline{\varphi}|S_1 = \varphi$.

Proof. This follows from (2.1) by taking $S' = R_2$ and $\alpha = x \in R_2$.

(2.3) A left $S$-module $M$ can be converted to a left-$R$-module if
and only if there exists an \( f \in \text{Hom}_Z(M, M) \) such that \( f(s.m) - s.f(m) = ds.m \), for every \( s \in S, m \in M \).

**Proof.** If \( M \) is an \( R \)-module we may take \( f \in \text{Hom}_Z(M, M) \) defined by \( f(m) = x.m \). The converse follows from (2.1) by taking

\[
S' = \text{Hom}_Z(M, M), \quad \alpha = f \quad \text{and} \quad \varphi : S \rightarrow S'
\]
to be the mapping which defines the \( S \)-module structure on \( M \).

(2.4) If \( M \) is a projective left \( S \)-module, then \( M \) can be converted into a left \( R \)-module.

**Proof.** We first remark that \( S \) can be considered as a left \( R \)-module. In fact, with the notation of (2.3) we choose \( f = d \in \text{Hom}_Z(S, S) \). By a direct sum argument, it is clear that any free left \( S \)-module can be regarded as an \( R \)-module. Now let \( M \) be any projective left \( S \)-module and let \( M \) be a direct summand of a free \( S \)-module \( F \). Since \( F \) is a left \( R \)-module, there exists an \( f \in \text{Hom}_Z(F, F) \) such that \( f(s.m) - s.f(m) = ds.m; s \in S, m \in F \). Let \( p: F \rightarrow M \) be an \( S \)-projection of \( F \) on \( M \). It is easily seen that \( g = f \circ p \big|_M \) satisfies \( g(s.m) - s.g(m) = ds.m \). Hence \( M \) can be regarded as an \( R \)-module.

(2.5) \( R \) becomes a filtered ring by setting \( F_p R = \sum_{p \leq i \leq p'} S.x^i \). The associated graded ring \( E^o(R) \) of \( R \) is isomorphic to \( S[x] \), the usual polynomial ring in one variable \( x \) over \( S \).

**Proof.** See [3].

3. Lifting of resolutions. Let \( M \) be a left \( R \)-module and let

\[
\cdots \rightarrow X_i \xrightarrow{d_i} X_{i-1} \rightarrow \cdots \rightarrow X_0 \xrightarrow{\varepsilon} M \rightarrow 0
\]

be an \( S \)-projective resolution of \( M \). Our aim in this section is to construct an \( R \)-projective resolution which "lifts" the above resolution.

We first prove the following

**Lemma.** There exist \( f_i \in \text{Hom}_Z(X_i, X_i) \) such that

(i) \( f_i(s.\alpha) - s.f_i(\alpha) = ds.\alpha \) for \( s \in S, \alpha \in X_i \);

(ii) \( d_i \circ f_i = f_{i-1} \circ d_{i-1}, i \geq 1, \) and \( \varepsilon \circ f_0 = f \circ \varepsilon \),

where \( f \in \text{Hom}_Z(M, M) \) is the mapping given by \( f(m) = x.m \).

**Proof.** Since \( X_0 \) is \( S \)-projective, it follows from (2.4) and (2.3) that there exists an \( f''_0 \in \text{Hom}_Z(X_0, X_0) \) such that \( f''_0(s.\alpha) - s.f''_0(\alpha) = ds.\alpha \) for \( s \in S, \alpha \in X_0 \). The map \( \varepsilon \circ f''_0 - f''_0 \varepsilon : X_0 \rightarrow M \) is easily verified to be \( S \)-linear. Since \( X_0 \) is \( S \)-projective there exists an \( f''_0' \in \text{Hom}_S(X_0, X_0) \)
such that $\varepsilon \circ f'_0 - f \circ \varepsilon = \varepsilon \circ f''_0$. We choose $f_0 = f'_0 - f''_0$. Then (i) and (ii) are verified for $i = 0$.

Assume inductively that $f_i$ for $0 \leq j \leq i - 1$ have already been defined satisfying (i) and (ii). Since $X_i$ is $S$-projective, there exists $f'_i \in \text{Hom}_S(X_i, X_i)$ such that $f'_i(s\alpha) - sf'_i(\alpha) = ds\alpha$ for $s \in S, \alpha \in X_i$. The map $d_i \circ f'_i - f_{i-1} \circ d_i : X_i \to X_{i-1}$ is easily verified to be $S$-linear. We have, (with the convention $f_0 = f$ and $d_0 = \varepsilon$),

$$d_{i-1}(d_i \circ f'_i - f_{i-1} \circ d_i) = -d_{i-1} \circ f_{i-1} \circ d_i$$

$$= -f_{i-2} \circ d_{i-2} \circ d_i \quad \text{(by induction)}$$

$$= 0.
$$

Hence the image of $X_i$ by $d_i \circ f'_i - f_{i-1} \circ d_i$ is contained in the kernel of $d_{i-1} = \text{Im.} d_i$. Since $X_i$ is $S$-projective, there exists $f''_i \in \text{Hom}_S(X_i, X_i)$ such that $d_i \circ f'_i - f_{i-1} \circ d_i = d_i \circ f''_i$. We may choose $f_i = f'_i - f''_i$ and $f_i$ satisfies (i) and (ii). This completes the proof of the lemma.

We set $X_{-1} = 0$ and define for $i \geq 0$

$$\bar{X} = R \otimes_S X_i + Ry \otimes_S X_{i-1},$$

where $y$ is a dummy. We set $d_0 = 0$ and define for $i \geq 1$, the $R$-homomorphism $\bar{d}_i : \bar{X}_i \to \bar{X}_{i-1}$ by

$$\bar{d}_i(1 \otimes \alpha') = 1 \otimes d_i \alpha, \alpha \in X_i$$

and

$$\bar{d}_i(y \otimes \alpha') = y \otimes d_{i-1} \alpha' + (-1)^{i-1} x \otimes \alpha' + (-1)^i 1 \otimes f_{i-1}(\alpha'), \alpha' \in X_{i-1}.$$

We define the $R$-homomorphism $\bar{\varepsilon} : \bar{X}_0 = R \otimes_S X_0 \to M$ by

$$\bar{\varepsilon}(1 \otimes \alpha) = \varepsilon(\alpha), \alpha \in X_0.$$

**Theorem 1.** The sequence

$$(*) \quad \cdots \to \bar{X}_i \xrightarrow{\bar{d}_i} \bar{X}_{i-1} \to \cdots \to \bar{X}_0 \xrightarrow{\bar{\varepsilon}} M \to 0$$

is an $R$-projective resolution of $M$.

**Proof.** For $\alpha \in X_i$, $\bar{\varepsilon} \circ \bar{d}_i(1 \otimes \alpha) = \bar{\varepsilon}(1 \otimes d_i \alpha) = \varepsilon d_i(\alpha) = 0$, and for

$$\alpha' \in X_0, \bar{\varepsilon} \circ \bar{d}_i(y \otimes \alpha') = \bar{\varepsilon}(x \otimes \alpha' - 1 \otimes f_i(\alpha'))$$

$$= f \circ \varepsilon(\alpha') - \varepsilon \circ f_0(\alpha') = 0.$$

For $i \geq 1$, we have

$$\bar{d}_{i-1} \circ \bar{d}_i(1 \otimes \alpha) = 1 \otimes d_{i-1} \circ d_i \alpha = 0, \alpha \in X_i,$$

and
Thus \((*)\) is a complex of left \(R\)-modules. To prove that the complex is acyclic, we define a suitable filtration on the complex whose associated graded is acyclic. By a well-known lemma on filtered complexes the acyclicity of \((*)\) follows immediately. For \(i \geq 0\), let

\[ F_p \tilde{X}_i = F_p R \otimes \tilde{X}_i + F_{p-1} R \cdot y \otimes \tilde{X}_{i-1}, \]

where \(\{F_p R\}\) is the filtration on \(R\) defined in (2.5). We define

\[ F_p M = M \]

for every \(p\).

It is easily seen that \(\{F_p X_i\}\) defines a filtration on \(\tilde{X}_i\) and that \(\tilde{d}_i(F_p \tilde{X}_i) \subset F_p \tilde{X}_{i-1}\) for \(i \geq 1\) and \(\varepsilon(F_p X_0) \subset F_p M\). We thus get for \(p \geq 0\) the complex

\[ \cdots \rightarrow E_p^0(\tilde{X}_i) \rightarrow E_p^0(\tilde{X}_{i-1}) \rightarrow \cdots \rightarrow E_p^0(\tilde{X}_1) \rightarrow E_p^0(M) \rightarrow 0. \]

We note that \(E_p^0(M) = 0\) for \(p \neq 0\) and \(E_0^0(M) = M\).

Let \(S[x]\) denote the polynomial ring in one variable \(x\) over \(S\). We regard \(M\) as an \(S[x]\)-module by setting \(xM = 0\). We set \(X'_1 = 0\) and define \(X'_i\) for \(i \geq 0\) by

\[ X'_i = S_p[x] \otimes X_i + S_{p-1}[x] \cdot y \otimes \tilde{X}_{i-1}. \]

We set \(d'_0 = 0\) and for \(i \geq 1\) define the left \(S[x]\)-homomorphism \(d'_i: X'_i \rightarrow X'_{i-1}\) by

\[ d'_i(1 \otimes \alpha) = 1 \otimes d_i \alpha, \quad \alpha \in X_i, \]

\[ d'_i(y \otimes \alpha') = y \otimes d_{i-1} \alpha' + (-1)^{i-1} x \otimes \alpha', \quad \alpha' \in X_{i-1}. \]

We define the \(S[x]\)-homomorphism \(\varepsilon': X'_0 \rightarrow M\) by setting

\[ \varepsilon'(1 \otimes \alpha) = \varepsilon(\alpha). \]

It is easily verified [4, p. 210] that \((X'_i, d'_i)\) is a left \(S[x]\)-projective resolution for \(M\).

Let \(S_p[x]\) be the \(p^{th}\) homogeneous component of the usual gradation of \(S[x]\) given by powers of \(x\). We introduce a gradation on
$X_i'$ by setting

$$X_i'^p = S_p[x] \otimes X_i + S_{p-1}[y] \otimes X_{i-1}.$$  

We take the trivial gradation on $M$ i.e., $M^p = 0$ for $p > 0$ and $M^0 = M$. It is easily seen that $d_{i*}(X_i'^p) \subset X_i'^{p+1}$ and $\varepsilon'(X_i'^p) \subset M^p$ for every $p$. We thus get for every $p$ an exact sequence

$$\cdots \longrightarrow X_i'^p \xrightarrow{d_{i*}} X_i'^{p+1} \longrightarrow \cdots \longrightarrow X_0'^p \xrightarrow{\varepsilon'} M^p \longrightarrow 0.$$ 

Clearly $E_p(X_i') \approx X_i'^p$ and $E_p(M) \approx M^p$ for every $p$. Since for any $r \in F_{p-1}R$ and $\alpha' \in X_{i-1}$, we have $r \otimes f_{i-1}(\alpha') \in F_{p-1}X_{i-1}$, it follows that $E_p(\delta_i) = d_i'^p$. Since $\text{(**)}$ is exact, it follows that $(E_p'(X_i'), E_p'(d_i))$ is exact and hence $(\ast)$ is exact. Since $X_i$ is clearly $R$-projective, the theorem is proved.

4. The case of local rings. Our aim in this section is to prove the following.

**Proposition 1.** Let $S$ be a (commutative, Noetherian) local ring and let $\mathfrak{m}$ denote its unique maximal ideal. Let $d$ be a derivation of $S$ such that $d(S) \subset \mathfrak{m}$ and let $R = S[x, d]$. Then

$$\text{l.gl. dim } R = 1 + \text{gl. dim } S.$$ 

For proving this proposition, we need the following.

**Lemma.** Let $S$ be a commutative ring and let $M$ be an $R$-module. Suppose

$$0 \longrightarrow X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

is an $S$-projective resolution of $M$. Assume that the following conditions hold.

1. $X_n$ is $S$-free of rank 1.
2. There exists an $S$-module $N$ with $xN = 0$ and $\text{Ext}_R^1(M, N) \neq 0$.

Then $hd_R M = n + 1$.

**Proof.** Using the complex $(\ast)$ of Theorem 1, we find that $hd_R M \leq n + 1$. We now compute $\text{Ext}_R^{n+1}(M, N')$ for any $R$-module $N'$. We have

$$\text{Ext}_R^{n+1}(M, N') = \text{Hom}_S(X_n, N')/B^n,$$

where $B^n$ is the set of all $g \in \text{Hom}_S(X_n, N')$ such that there exist $g_1 \in \text{Hom}_S(X_n, N')$ and $g_2 \in \text{Hom}_S(X_{n-1}, N')$ with

$$g(\alpha) = g_1(d_n \alpha) + (-1)^{n-1} xg_1(\alpha) + (-1)^ng_2(\alpha)$$

for any $\alpha \in X_n$. 

Let \( \beta \) be a free generator of \( X_n \) as an \( S \)-module and let \( f_n(\beta) = s \beta; s \in S \). If \( g \in B^s \), we have

\[
g(\beta) = g_s(d_n \beta) + (-1)^{n-1}(x-s) g_1(\beta)
\]

Let \( \theta \) be the automorphism of \( R \) such that \( \theta(x) = x + s \) and \( \theta | S = \text{identity} \). (This exists in view of (2.1)). If we choose \( N' = e N \) (i.e., \( N \) considered as an \( R \)-module through \( \theta \)), we find \( g(\alpha) = g_s(d_n \alpha) \) for any \( \alpha \in X_n \). Thus, \( B^s = B^s_\theta = \{ g \in \text{Hom}_S(X_n, N') | g(\alpha) = g_s(d_n \alpha) \} \) for some \( g_s \in \text{Hom}_S(X_{n-1}, N') \) for every \( \alpha \in X_{n-1} \). However, using the resolution \( (X_i, d_i) \) for \( M \) to compute \( \text{Ext} \), we find \( \text{Ext}_S^z(M, N') \approx \text{Hom}_S(X_n, N')/B^s_\theta \). Hence

\[
\text{Ext}_R^{n+1}(M, N') \approx \text{Ext}_S^z(M, N') \approx \text{Ext}_S^z(M, N) \neq (0),
\]

since \( N \) and \( N' \) are isomorphic as \( S \)-modules. This proves the lemma.

**Proof of proposition.** By [2, p. 74, Prop. 2], it follows that \( \text{gl. dim } R \geq \text{gl. dim } S \). Thus, if \( \text{gl. dim } S = \infty \), we have \( \text{gl. dim } R = \infty \) and the proposition is proved. We therefore assume that \( \text{gl. dim } S = n < \infty \). If \( M = S/\mathfrak{m} \), we have \( \text{hd}_S M = n \). Let

\[
0 \longrightarrow X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0
\]

be the “Koszul resolution” for \( M \) [1, p. 151]. Since \( X_n = E_n(y_1, \ldots, y_n) \), where \( E_n(y_1, \ldots, y_n) \) is the \( n \)th component of the exterior algebra on \( y_1, \ldots, y_n \) over \( S \), condition (i) of the above lemma is satisfied. Since \( d(S) \subseteq \mathfrak{m} \), it is clear that \( M \) can be regarded as an \( R \)-module satisfying \( xM = 0 \) (See (2,3)). Since \( \text{Ext}_S^z(M, M) \neq (0) \), [1, p. 153], condition (2) of the lemma is satisfied with \( N = M \). Thus, by the above lemma, we have \( \text{hd}_n M = n + 1 \). Hence \( \text{gl. dim } R \geq n + 1 \). Since \( \text{gl. dim } R \leq n + 1 \) [6, Th. 1 or 3], the proposition is proved.

5. The case of Noetherian rings. In this section, we prove the following

**Theorem 2.** Let \( S \) be a commutative Noetherian ring and let \( d \) be a derivation of \( S \) such that any one of the following two conditions is satisfied:

1. \( d(S) \subseteq \text{Radical of } S \),
2. \( d(S) \) generates a proper ideal of \( S \) and \( \text{Krull dim } S/\mathfrak{m} \) is the same for all the maximal ideals \( \mathfrak{m} \) of \( S \).

If \( R = S[x, d] \), we have

\[
l. \text{gl. dim } R = 1 + \text{gl. dim } S.
\]
Proof. As in the proof of Proposition 1, we need only prove that 1. gl. dim \( R \geq 1 + \text{gl. dim} \ S \) assuming gl. dim \( S < \infty \). Since \( \text{gl. dim} \ S = \sup \text{gl. dim} \ S_{\mathfrak{M}} \) where \( \mathfrak{M} \) runs over all the maximal ideals of \( S \), it is clear that under either of the conditions of the theorem, there exists a maximal ideal \( \mathfrak{M} \) such that \( \text{gl. dim} \ S = \text{gl. dim} \ S_{\mathfrak{M}} \) and \( d(S) \subseteq \mathfrak{M} \).

The derivation \( d \) of \( S \) induces a derivation \( \bar{d} \) of \( S_{\mathfrak{M}} \) if we set

\[
\bar{d}(s) = \frac{ds - s' ds'}{s'^2}; \quad s, s' \in S, \quad s' \in \mathfrak{M}.
\]

It is clear that \( \bar{d}(S_{\mathfrak{M}}) \subseteq \mathfrak{M}S_{\mathfrak{M}} \). Hence by Proposition 1, § 4, we have

1. \( \text{gl. dim} \ S_{\mathfrak{M}} \{x, d\} = 1 + \text{gl. dim} \ S_{\mathfrak{M}} = 1 + \text{gl. dim} \ S \).

Thus, the theorem will be proved if we prove the following

**Lemma.** If \( \mathfrak{M} \) is any maximal ideal of \( S \), we have

1. \( \text{gl. dim} \ S_{\mathfrak{M}} \{x, d\} \geq 1, \text{gl. dim} \ S_{\mathfrak{M}} \{x, \bar{d}\} \).

**Proof of the lemma.** Let us set \( R = S\{x, d\} \) and \( \bar{R} = S_{\mathfrak{M}}\{x, \bar{d}\} \).

Let \( \eta: S \longrightarrow S_{\mathfrak{M}} \) denote the ring homomorphism defined by \( \eta(s) = \text{class of } s/1 \). Since \( \bar{d} \circ \eta = \eta \circ d \), \( \eta \) induces (see (2.2)) a ring homomorphism \( \bar{\eta}: R \longrightarrow \bar{R} \) such that \( \bar{\eta} | S = \eta \).

We first prove the following two statements:

1. \( \bar{R} \) is \( R \)-flat as a right \( R \)-module (through \( \bar{\eta} \)).
2. If \( M \) is any left \( \bar{R} \)-module, there exists a left \( R \)-module \( M' \) and a left \( \bar{R} \)-isomorphism \( M \cong \bar{R} \otimes_R M' \).

The left \( S_{\mathfrak{M}} \)-isomorphism \( \varphi: S_{\mathfrak{M}} \otimes_S R \longrightarrow \bar{R} \) given by \( \varphi(1 \otimes x') = x' \in \bar{R} \) satisfies \( \varphi(1 \otimes f) = \bar{\eta}(f) \) for any \( f \in R \). We have

\[
\varphi(1 \otimes fg) = \bar{\eta}(fg) = \bar{\eta}(f)\bar{\eta}(g) = \varphi(1 \otimes f)\bar{\eta}(g).
\]

Thus, \( \varphi \) is an isomorphism of right \( R \)-modules. Since \( S_{\mathfrak{M}} \otimes_S R \) is right \( R \)-flat, (1) is proved. Let

\[
\bar{F}_1 \xrightarrow{\lambda} \bar{F} \xrightarrow{\mu} M \longrightarrow 0
\]

be an exact sequence where \( \bar{F}_1 \) and \( \bar{F} \) are \( \bar{R} \)-free with bases \( \{e_a\} \) and \( \{f_{s}\} \) respectively. We then have

\[
\lambda(e_a) = \eta\left(\frac{1}{s_a}\right) \sum \frac{1}{\eta}(a_{ab})f_{s}; \quad a_{ab} \in R, \quad s_a \in S - \mathfrak{M}.
\]

Let \( \theta \) be the \( \bar{R} \)-automorphism of \( \bar{F}_1 \) defined by \( \theta(e_a) = \eta(s_a)e_a \). Let
\( \lambda' = \lambda \circ \theta \). We then have
\[
\lambda'(e_a) = \sum_{\beta} \frac{1}{\gamma_{ij}} (a_{\alpha\beta}) f_{\beta},
\]
and the sequence
\[
F'_1 \xrightarrow{\lambda'} F' \xrightarrow{\mu} M \rightarrow 0
\]
is exact. Let \( F'_1 \) (resp. \( F' \)) be the free \( R \)-module generated by \( \{e_a\} \) (resp. \( \{f_{\beta}\} \)) and let \( \lambda'' : F'_1 \rightarrow F' \) be the \( R \)-homomorphism defined by
\[
\lambda''(e_a) = \sum_{\beta} a_{\alpha\beta} f_{\beta}.
\]
It is easily seen that if we take \( M' = c \circ \ker \lambda'' \), we have \( M \cong R \otimes M' \).

This proves (2). We now complete the proof of the lemma.

Let \( M \) be any left \( R \)-module and let \( M' \) be a left \( R \)-module such that (2) is satisfied. Let
\[
\cdots \rightarrow X_n \xrightarrow{d_n} X_{n-1} \rightarrow \cdots \rightarrow X_0 \rightarrow M' \rightarrow 0
\]
be a resolution of \( M' \) as a left \( R \)-module. Then
\[
R \otimes_R X_n \xrightarrow{\otimes d_n} R \otimes_R X_{n-1} \rightarrow \cdots \rightarrow R \otimes_R X_0 \rightarrow M \rightarrow 0
\]
is exact in view of (1). Since \( R \otimes_R X_i \) is \( R \)-projective, it follows that
\((R \otimes_R X_i, 1 \otimes d_i)\) is an \( R \)-projective resolution of \( M \). In particular, we have \( \text{hd}(M') \leq \text{hd}(M) \leq \text{gl}\ dim R \). Since \( M \) is arbitrary, it follows that \( \text{gl} \ dim R \leq \text{gl} \ dim R \). This proves the lemma and hence the theorem.

**Remark.** Let \( S = K[x_1, \ldots, x_n] \) be the polynomial ring in \( n \) variables over a field \( K \). It is well-known [7, Chap. III Cor. 4 to Th. 5] that Krull \( \text{dim} S_{\mathfrak{m}} \) is the same for all maximal ideals \( \mathfrak{m} \) of \( S \). Let \( d \) be a \( K \)-derivation of \( S \) given by \( d(x_i) = f_i \). Then the derivation \( d \) satisfies condition (2) of Theorem 2 if and only if \( f_i, 1 \leq i \leq n \) are not coprime and in this case we may apply the theorem and we have \( \text{gl} \ dim R = n + 1 \). This includes the special case of Theorem 1 of [6] in which \( K \) is a field.

**References**


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