Graded Witt Ring and Unramified Cohomology

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Abstract. It is proved that for a smooth affine curve $X$ over a local ring or global field, the graded Witt ring of $X$ is isomorphic to the graded unramified cohomology ring of $X$. If $X$ is projective and has a rational point, the same result holds if and only if every quadratic space defined on the complement of a rational point extends to $X$. Such an extension is possible, for instance, if the canonical line bundle on $X$ is a square in $\text{Pic } X$.

Key words. Witt ring, unramified étale cohomology, signature, real spectrum, cohomological dimension.

0. Introduction

Let $k$ be a field of characteristic different from 2 and $X$ a smooth variety over $k$. Let $k(X)$ denote the function field of $X$. Let $\partial: W(k(X)) \rightarrow \bigoplus_{x \in X^1} W(k(x))$ denote the map induced by the second residue homomorphism for a choice of local parameters at codimension one points $X^1$ of $X$. The kernel of $\partial$ is independent of the choice of the local parameters and is defined to be the unramified Witt group of $X$ (cf. [CTPa]) and denoted by $W_{nr}(X)$. In view of results of Pardon [P] and Barge–Sansuc–Vogel, $W_{nr}(X)$ consists of classes of forms over $k(X)$ which extend, at every point $x$ of $X^1$, to a nonsingular quadratic space over the local ring $\mathcal{O}_{X,x}$. The groups

$$I_n(X) = I^n(k(X)) \cap W_{nr}(X), \quad n \geq 0, \quad I^n(k(X))$$

denoting the $n$th power of the fundamental ideal $I(k(X))$, give a filtration on $W_{nr}(X)$. The aim of this paper is to compare the associated graded unramified Witt ring $\bigoplus_{n \geq 0} I_n(X)/I_{n+1}(X)$ with the graded unramified cohomology ring $\bigoplus_{n \geq 0} H^0(X, \mathcal{K}^n)$ (cf. [BO]) where $\mathcal{K}^n$ denotes the Zariski sheaf associated to the presheaf $U \mapsto H^n_{et}(U, \mu_2)$.

If $X = \text{Spec } k$, and $P_n(k)$ denotes the set of $n$-fold Pfister forms over $k$, Arason [A] proves that there is a well defined map $e_n: P_n(k) \rightarrow H^n(k)$, such that $e_n(\langle a_1, a_2, \ldots, a_n \rangle) = (a_1) \cup (a_2) \cup \cdots \cup (a_n), (a) \in H^1(k)$ denoting the square class of $a \in k^*$ (here we abbreviate $H^n(k) = H^n_{et}(\text{Spec } k, \mu_2)$). For a general field $k$, it is an open question whether $e_n$ extends to a homomorphism $I^n(k) \rightarrow H^n(k)$ and if so, whether it
is surjective with kernel $I^{n+1}(k)$. This question has an affirmative answer for local and global fields as well as for function fields in one variable over a local or global field (cf. [AEJ 2]). We call a variety $X$ good if for every point $x \in X$ of codimension $\leq 1$, the graded Witt ring of $k(x)$ is isomorphic to the graded cohomology ring of $k(x)$. Examples of good varieties are provided by curves over local and global fields and varieties of dimension $\leq 3$ over $\mathbb{R}$ or $\mathbb{C}$ ([AEJ 2] and [AEJ 1]). If $X$ is good, we have a well defined homomorphism

$$e_n: I_n(X) \to H^0(X, \mathcal{H}^n)$$

with kernel $I_{n+1}(X)$ (cf. [Pa]). We thus have an induced homomorphism

$$e = \{e_n\}: \bigoplus_{n \geq 0} I_n(X)/I_{n+1}(X) \to \bigoplus_{n \geq 0} H^0(X, \mathcal{H}^n)$$

of the graded unramified Witt ring of $X$ into the graded unramified cohomology ring of $X$.

**Question (Q).** If $X$ is good, is the map $e$ an isomorphism?

The above question may be thought of as a global version of the open question on quadratic forms over fields, stated above. In this paper, we answer this question in the affirmative, for smooth affine curves over a local or a global field. The result is true for projective curves $X$ as well if the canonical line bundle $\Omega_X$ is a square in $\text{Pic } X$. The line bundle $\Omega_X$ is a square in $\text{Pic } X$ if $X$ is a hyperelliptic curve of genus $g \geq 2$ with a rational point of ramification over $\mathbb{P}^1$ or if $X$ is an elliptic curve. We show that $\Omega_X$ is a square in $\text{Pic } X$, if all the 2-torsion points of $\text{Pic } X_{\overline{k}}$ are defined over $k$, $\overline{k}$ denoting the separable closure of $k$. The condition $\Omega_X$ is a square in $\text{Pic } X$ is imposed to ensure that every quadratic space over $X \setminus \{x_0\}$, $x_0$ a rational point of $X$ extends to $X$ (cf. [GHKS]).

The following question seems to be of independent interest: Does a smooth projective curve $X$ over $k$ have ‘extension property’ for quadratic spaces, namely: There exists $x_0 \in X(k)$ such that every quadratic space over $X \setminus \{x_0\}$ extends to $X$. In fact, we prove that for a local field $k$ and a smooth projective curve $X$ over $k$, with a rational point, the map $e_2: I_2(X) \to H^0(X, \mathcal{H}^2)$ is surjective if and only if $X$ has an extension property for quadratic spaces. It is shown in [PaSc] that the extension property for hyperelliptic curves over local fields is equivalent to $\Omega_X$ being a square in $\text{Pic } X$. Thus, in view of the examples in [PaSc], one sees that the map $e_2$ is not in general surjective, even for smooth projective curves over a local field. Thus, Merkurjev’s theorem does not admit a globalisation to curves.

We exploit a construction of the ‘mod-2 signature’ homomorphism on the unramified cohomology defined in [CTPa] to prove the main theorem. We also use results of Kato on two-dimensional class field theory.
1. Some Known Results

In this section, we list a few results which we shall use in the paper. We denote by $k$ a field with $\text{char } k \neq 2$.

(1.1) Let $X$ be a smooth irreducible variety over $k$. We abbreviate $H^n_{et}(X, \mu_2) = H^n(X)$. Let $l = k[X]/(X^2 + a)$ be a quadratic extension (we include the case $l \cong k \times k$) and $\pi: \text{Spec } l \to \text{Spec } k$ the projection. Let $X_1 = X \times_{\text{Spec } k} \text{Spec } l$ and $\chi_a \in H^1(k)$ the character corresponding to the element $a \in k^*/k^{*2}$. The short exact sequence of étale sheaves

$$0 \to (\mu_2)_k \to \pi_*(\mu_2)_l \to (\mu_2)_k \to 0$$

yields a long exact sequence in étale cohomology:

$$H^n(X) \xrightarrow{\cup \chi_a} H^{n+1}(X) \xrightarrow{i} H^{n+1}(X_1) \xrightarrow{s} H^{n+1}(X).$$

(1.2) Let $l/k$ be as above. Let $s: W(l) \to W(k)$ be the Scharlau transfer map (cf. [L], p. 169) and $\langle -a \rangle: W(k) \to W(k)$ the map obtained by tensoring with $\langle -a \rangle = \langle 1, -a \rangle$. Whenever Spec $k$ and Spec $l$ are good, there exists a long exact sequence of Witt groups compatible with the long exact sequence in cohomology defined in (1.1) through the maps $e_n$ (cf. [AEJ 1]).

$$I^n(k) \langle -a \rangle \to I^{n+1}(k) \xrightarrow{i} I^{n+1}(l) \xrightarrow{s} I^{n+1}(k)$$

(1.3) Let $X$ be a smooth integral curve over $k$. There is a residue homomorphism

$$\partial: H^n(k(X)) \to \bigoplus_{x \in X} H^{n-1}(k(x)),$$

(by $x \in X$, for a curve $X$, we mean $x$ running over the set of closed points of $X$) with $\ker \partial \cong H^0(X, \mathcal{O}_X^n)$, $\text{coker } \partial \cong H^1(X, \mathcal{O}_X^n)$. The group $H^1(X, \mathcal{O}_X^n)$ is the subgroup of $H^{n+1}(X)$ defined by elements whose restriction to a non-empty open set is trivial. Further, the sequence

$$0 \to H^1(X, \mathcal{O}_X^n) \to H^{n+1}(X) \to H^0(X, \mathcal{O}_X^{n+1}) \to 0$$

is exact ([B], p. 85).

(1.4) Let $X$ be any smooth variety over $k$ which is good. For a codimension one point $x \in X^1$, let

$$\partial_x: I^n(k(X)) \to I^{n-1}(k(x))$$
denote the second residue homomorphism with respect to a choice of a parameter at \( x \). Then, the diagram

\[
\begin{array}{ccc}
I^n(k(X)) & \xrightarrow{\partial} & \bigoplus_{x \in X^1} I^{n-1}(k(x)) \\
\downarrow e_n & & \downarrow (e_{n-1}) \\
H^n(k(X)) & \xrightarrow{\partial} & \bigoplus_{x \in X^1} H^{n-1}(k(x))
\end{array}
\]

is commutative and induces maps \( e_n: I_n(X) \to H^0(X, \mathcal{H}^n) \) (cf. [Pa]).

(1.5) We recall the following results from [AEJ 2], part of which rely on Kato's results in [Ka]. Let \( X \) be a smooth integral curve over \( k \). If \( k \) is a non-Archimedean local field, for \( n \geq 4, I^n(k(X)) = 0, H^n(k(X)) = 0 \) and \( e_3: I^3(k(X)) \to H^3(k(X)) \) is an isomorphism. If \( k \) is a global field which is totally imaginary, the injectivity of \( I^3(k(X)) \to \bigoplus_{v} I^3(k_v(X)), \) \( v \) running over all places of \( k \), yields that \( I^4(k(X)) = 0, H^4(k(X)) = 0 \) and \( e_3: I^3(k(X)) \cong H^3(k(X)) \) is an isomorphism ([AEJ 2]).

(1.6) The map \( e_0: W(X) \to H^0(X, \mathcal{H}^0) \cong \mathbb{Z}/2 \) is the rank homomorphism and \( e_1: I(X) \to H^0(X, \mathcal{H}^1) = H^1(X) \) is the discriminant homomorphism. These are always surjective. Thus, to answer (Q) positively for any good variety \( X \), one needs to prove the surjectivity of \( e_n: I_n(X) \to H^0(X, \mathcal{H}^n) \) for \( n \geq 2 \).

2. An Extension Lemma

In this section, we prove the following:

PROPOSITION 2.1. Let \( X \) be a smooth projective curve over a field \( k \) with \( \text{char } k \neq 2 \). Let \( x_0 \) be a \( k \)-rational point of \( X \). Suppose every element in the 2-torsion of \( \text{Pic} X_k, k \) denoting the separable closure of \( k \), is defined over \( k \). Then, every quadratic space over \( X_k \backslash \{x_0\} \) extends to \( X \).

In view of a residue theorem of [GHKS], it suffices to prove the following proposition.

PROPOSITION 2.2. Let \( X \) be as in (2.1) and let \( S(X_k) \) denote the set of all square roots of the canonical bundle \( \Omega_X \) in \( \text{Pic} X_k \). Then \( S(X_k) \) is defined over \( k \).

Proof. We assume without loss of generality, that genus \( X \geq 2 \). Then \( S(X_k) \) is a principal homogeneous space over \( \text{Pic} X_k \cong H^1(X_k) \). The function \( \varphi: S(X_k) \to \mathbb{F}_2 \) defined by \( \varphi(\mathcal{L}) = \text{dim } H^0(X_k, \mathcal{L}) \mod 2 \) is a quadratic function on \( S(X_k) \) (in the sense of [At], p. 48) which has a nontrivial zero, since genus \( X \geq 1 \) (cf. [Mu]). Let \( a \in S(X_k) \) be such that \( \varphi(a) = 0 \). Then, the map \( \varphi_a: \text{Pic} X_k \to \mathbb{F}_2 \) defined by \( \varphi_a(x) = \varphi(a + x) \) is a nondegenerate quadratic form on the \( \mathbb{F}_2 \)-vector space \( \text{Pic} X_k \).
whose associated bilinear form is simply the cup-product on $H^1(X_k, \mu_2)$ (cf. [Mu]).

We transform the Galois action of $G(k/k)$ on $S(X_k)$ into an affine action on $\text{Pic}_2(X_k)$ through the map $S(X_k) \cong \text{Pic}_2(X_k)$, $x \mapsto x - a$. More explicitly, for $\sigma \in G(k/k)$, $x \in \text{Pic}_2(X_k), x^\sigma = \sigma x + \sigma a - a = x + \sigma a - a$, since $G(k/k)$ acts trivially on $\text{Pic}_2(X_k)$, by assumption. Since $\dim H^0(X_k, \mathcal{L}) = \dim H^0(X_k, \sigma \mathcal{L})$ for any $\mathcal{L} \in \text{Pic}_2(X_k), \sigma \in G(k/k)$, $\sigma$ preserves the quadratic function $\phi_n$ on $\text{Pic}_2(X_k)$. By a lemma of Serre quoted in ([At], Lemma 5.1), $\sigma$ has a fixed point, i.e. for some $x \in \text{Pic}_2(X_k), x = x^\sigma = x + \sigma a - a$; i.e., $\sigma a = a$ and, hence, $\sigma(a + x) = a + x$, for each $x \in \text{Pic}_2(X_k)$. Thus, $\sigma$ acts trivially on $S(X_k)$; i.e. $S(X_k)^G = S(X_k)$. Since $X(k) \neq \emptyset$, Pic $X = (\text{Pic}_2(X_k))^G(k)$ and every element of $S(X_k)$ is defined over $k$. This completes the proof of (2.2).

### 3. Fields with Finite 2-Cohomological Dimension

Let $Z$ be a smooth integral curve over a field $k$. Throughout, we assume that $\text{char } k \neq 2$ and $Z$ is good. We define a homomorphism

$$\phi_n : H^0(Z, \mathcal{H}^n) \to H^1(Z, \mathcal{H}^{n+1})$$

whose kernel contains $e_n(I_n(Z))$. In other words,

$$I_n(Z) \xrightarrow{e_n} H^0(Z, \mathcal{H}^n) \xrightarrow{\phi_n} H^1(Z, \mathcal{H}^{n+1})$$

(*n)

is a complex. We also give a set of sufficient conditions under which (*n) is exact.

Let $x \in H^0(Z, \mathcal{H}^n)$. Since $Z$ is good, the map $e_n : I^n(k(Z)) \to H^n(k(Z))$ is surjective so that there exists $r \in I^n(k(Z))$ such that $e_n(r) = x$. Since

$$e_{n-1} \circ \delta(r) = \delta \circ e_n(r) = \delta(x) = 0,$$

we have, $\delta(r) \in \bigoplus_{x \in Z} I^n(k(x))$. Let the map $\delta_n$ be defined by the exact sequence

$$H^{n+1}(k(Z)) \xrightarrow{\delta} \bigoplus_{x \in Z} H^n(k(x)) \xrightarrow{\delta_n} H^1(Z, \mathcal{H}^{n+1}) \to 0,$$

i.e., $\delta_n$ identifies $H^1(Z, \mathcal{H}^{n+1})$ as the cokernel of $\delta$ (cf. (1.3)). Set $\phi_n(x) = \delta_n \circ e_n \circ \delta(r)$.

We verify that $\phi_n$ is well-defined. In fact, if $r_1, r_2 \in I^n(k(Z))$ are such that $e_n(r_1) = e_n(r_2) = x$, then, $r_{12} = r_1 - r_2$ belongs to $I^{n+1}(k(Z))$, $e_{n+1}(r_{12}) \in H^{n+1}(k(Z))$ and

$$\delta_n \circ e_n \circ \delta(r_{12}) = \delta_n \circ \delta \circ e_{n+1}(r_{12})$$

$$= 0.$$

Thus $\phi_n$ is a well-defined homomorphism. For $\beta \in I_n(Z)$, $\phi_n \circ e_n(\beta) = \delta_n \circ e_n \circ \delta(\beta) = 0$ so that (*n) is a complex. Further, from the definition of $\phi_n$, it follows that it commutes with $\bigcup_{X_{-1}}$; i.e., the diagram

$$\begin{array}{ccc}
H^0(Z, \mathcal{H}^n) & \xrightarrow{\phi_n} & H^1(Z, \mathcal{H}^{n+1}) \\
\bigcup_{X_{-1}} & & \bigcup_{X_{-1}} \\
H^0(Z, \mathcal{H}^{n+1}) & \xrightarrow{\phi_{n+1}} & H^1(Z, \mathcal{H}^{n+2})
\end{array}$$
is commutative, where $\cup_{\chi-1}$ on the right is the restriction of the map $\cup_{\chi-1}: H^{n+2}(Z) \to H^{n+3}(Z)$.

We define the condition $H(n)$ for a curve $Z$ as follows:

$$H(n):$$ The map $\cup_{\chi-1}: H^0(Z, \mathcal{H}^n) \to H^0(Z, \mathcal{H}^{n+1})$ is surjective.

Let $l = k[X]/(X^2 + 1)$. Let $l'$ be the extension of $k$ by adjoining $\sqrt{-1}$. Then either $l' = l$ or $l' = k$.

We assume in the rest of the section that $Z_t$ is also good.

**Lemma 3.1.** Suppose $Z$ satisfies $H(n)$. Then the sequence

$$H^1(Z, \mathcal{H}^n) \overset{i}{\to} H^1(Z_t, \mathcal{H}^n) \overset{s}{\to} H^1(Z, \mathcal{H}^n)$$

is exact, where $i$ and $s$ are the restrictions of the map $i: H^{n+1}(Z) \to H^{n+1}(Z_t)$ and $s: H^{n+1}(Z_t) \to H^{n+1}(Z)$ (cf. (1.1)).

**Proof.** By (1.3), we have the following commutative diagram with exact columns:

```
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
H^1(Z, \mathcal{H}^n) & H^1(Z_t, \mathcal{H}^n) & H^1(Z, \mathcal{H}^n) \\
\downarrow & \downarrow & \downarrow \\
H^*(Z) & H^{n+1}(Z) & H^{n+1}(Z_t) \\
\downarrow & \downarrow & \downarrow \\
H^0(Z, \mathcal{H}^n) & \cup_{\chi-1} H^0(Z, \mathcal{H}^{n+1}) \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
```

Exactness of the middle row (cf. (1.1)) and the surjectivity of the map $H^0(Z, \mathcal{H}^n) \overset{\cup_{\chi-1}}{\to} H^0(Z, \mathcal{H}^{n+1})$ imply the exactness of the top row.

**Lemma 3.2.** Let $\text{cd}_2(l') \leq n$ and let $\alpha \in H^0(Z, \mathcal{H}^n)$ be such that $\alpha \cup_{\chi-1} \in e_{n+1}(I_{n+1}(Z))$. Then there exists an element $r \in I^n(k(Z))$ such that $e_n(r) = \alpha$ and $\partial(r) \otimes \langle 1 \rangle = 0$.

**Proof.** If $l' = l$, any lift $r$ will do, since $\langle 1 \rangle = 0$ in $H^1(k)$ and $\partial(r) \otimes \langle 1 \rangle = 0$.

We therefore assume that $l' = l$. Since $\text{cd}_2(l) \leq n, \text{cd}_2(l(Z)) \leq n + 1$ so that $H^{n+2}(l(Z)) = 0$. Since $Z_t$ is good, $e_{n+2}: I^{n+2}(l(Z)) \to H^{n+2}(l(Z))$ is onto with kernel $I^{n+3}(l(Z))$. Thus, $I^{n+2}(l(Z)) = \bigcap_{k \geq 2} I^{n+k}(l(Z)) = 0$, by a theorem of Arason-Pfister ([L], p. 290). Let $\beta \in I_{n+1}(Z)$ be such that $e_{n+1}(\beta) = \alpha \cup_{\chi-1}$. Then, $e_{n+1}(\beta_l) = (\alpha \cup_{\chi-1})_l = 0$, by (1.2), suffix $l$ denoting images in the corresponding
groups over $Z$. Thus, $\beta_1 \in I^{n+2}(l(Z)) = 0$. Therefore, by (1.2), there exists $r_1 \in I^n(k(Z))$ such that $r_1 \otimes \langle 1 \rangle = \beta$. We have

$$\begin{align*}
(e_n(r_1) - x) \cup \chi_{-1} &= e_{n+1}(r_1 \otimes \langle 1 \rangle) - x \cup \chi_{-1} \\
&= e_{n+1}(\beta) - x \cup \chi_{-1} \\
&= 0.
\end{align*}$$

Thus, there exists $\eta \in H^n(l(Z))$ such that $s(\eta) = e_n(r_1) - x$. Let $e_n(\eta_1) = \eta$ and $r = r_1 - s(\eta_1)$. Then, $s(\eta_1) \otimes \langle 1 \rangle = 0$ by (1.2) and

$$\begin{align*}
\partial(r) \otimes \langle 1 \rangle &= \partial(r_1 \otimes \langle 1 \rangle) - \partial(s(\eta_1)) \otimes \langle 1 \rangle \\
&= \partial(\beta) \\
&= 0.
\end{align*}$$

Let $e_n(r) = e_n(r_1) = e_n(\eta) = e_n(r_1) - s \circ e_n(\eta_1) = e_n(r_1) - s(\eta) = x$.

**Proposition 3.3.** Suppose $c_d(l') \leq n$.

(A) If $e_{n+1}: I_{n+1}(Z) \to H^0(Z, \mathcal{H}^{n+1})$ is surjective, then, image $\phi_n \subset \text{image}(s: H^1(Z, \mathcal{H}^{n+1}) \to H^1(Z, \mathcal{H}^{n+1}))$.

(B) If $Z$ satisfies $H(n + 1)$ and $(*n + 1)$ is exact, then $(*n)$ is exact.

**Proof.** (A) Let $x \in H^0(Z, \mathcal{H}^n)$. Since $e_{n+1}$ is surjective, $x \cup \chi_{-1} \in e_{n+1}(I_{n+1}(Z))$. By (3.2), there exists $r \in I^n(k(Z))$ with $e_n(r) = x$ and $\partial(r) \otimes \langle 1 \rangle = 0$. Since

$$e_{n-1} \circ \partial(r) = \partial \circ e_n(r) = \partial(x) = 0, \quad \partial(r) \in \bigoplus_{x \in Z} I^n(k(x))$$

and $\partial(r) \otimes \langle 1 \rangle = 0$ implies that $\partial(r) = s(\eta)$ for some $\eta \in \bigoplus_{x \in Z} I^n(l(x))$. We have,

$$\begin{align*}
\phi_n(x) &= \delta_n \circ e_n \circ \partial(r) \\
&= \delta_n \circ e_n \circ s(\eta) \\
&= s \circ \delta_n \circ e_n(\eta)
\end{align*}$$

with $\delta_n \circ e_n(\eta) \in H^1(Z, \mathcal{H}^{n+1})$. The fact that $s$ commutes with $e_n$ and $\delta_n$ follows from ([A], 4.11 and 4.14).

(B) Let $x \in H^0(Z, \mathcal{H}^n)$ with $\phi_n(x) = 0$. We have

$$\phi_{n+1}(x \cup \chi_{-1}) = \phi_n(x) \cup \chi_{-1} = 0$$

and $(*n + 1)$ being exact, $x \cup \chi_{-1} \in e_{n+1}(I_{n+1}(Z))$. As in the proof of (A), one obtains $r \in I^n(k(Z))$ with $e_n(r) = x$, $\partial(r) \otimes \langle 1 \rangle = 0$ and an $\eta \in \bigoplus_{x \in Z} I^n(l(x))$.
such that \( s(\eta) = \partial r \). We have the following commutative diagram with exact rows.

\[
\begin{array}{cccccc}
H^{n+1}(k(Z)) & \overset{\delta}{\longrightarrow} & \bigoplus_{x \in Z} H^n(k(x)) & \overset{\delta_s}{\longrightarrow} & H^1(Z, \mathcal{X}^{n+1}) & \longrightarrow 0 \\
\downarrow i & & \downarrow i & & \downarrow i \\
H^{n+1}(l(Z)) & \overset{\delta}{\longrightarrow} & \bigoplus_{x \in Z} H^n(l(x)) & \overset{\delta_s}{\longrightarrow} & H^1(Z, \mathcal{X}^{n+1}) & \longrightarrow 0 \\
\downarrow s & & \downarrow s & & \downarrow s \\
H^{n+1}(k(Z)) & \overset{\delta}{\longrightarrow} & \bigoplus_{x \in Z} H^n(k(x)) & \overset{\delta_s}{\longrightarrow} & H^1(Z, \mathcal{X}^{n+1}) & \longrightarrow 0.
\end{array}
\]

We have

\[
s \circ \delta_n \circ e_n(\eta) = \delta_n \circ e_n(\partial(r)) \\
= \phi_n(x) \\
= 0.
\]

Since \( Z \) satisfies \( H(n+1) \), by (3.1), there exists \( \theta \in \bigoplus_{x \in Z} H^n(k(x)) \) such that \( \delta_n \circ e_n(\eta) = \delta_n \circ i(\theta) \). Exactness of the middle row gives an \( \eta_1 \in H^{n+1}(l(Z)) \) with \( \partial(\eta_1) = e_n(\eta) - i(\theta) \). Let \( \eta_2 \in I^{n+1}(l(Z)) \) be such that \( e_{n+1}(\eta_2) = \eta_1 \). We have, \( e_{n-1}(\partial(r - s(\eta_2))) = \partial e_n(r) - \partial e_n(s(\eta_2)) = \partial(x) = 0 \). Thus \( \partial(r - s(\eta_2)) \in \bigoplus_{x \in Z} I^n(k(x)) \).

Further,

\[
e_n(\partial(r - s(\eta_2))) = e_n \circ \partial(r) - \partial \circ e_{n+1} \circ s(\eta_2) \\
= e_n \circ \partial(r) - \partial \circ s(\eta_1) \\
= e_n \circ \partial(r) - s(e_n(\eta) - i(\theta)) \\
= e_n \circ \partial(r) - s \circ e_n(\eta) \\
= 0.
\]

Thus, \( \partial(r - s(\eta_2)) \in \bigoplus_{x \in Z} I^{n+1}(k(x)) \). Further, by (1.2),

\[
\partial(r - s(\eta_2)) \otimes \langle 1 \rangle = -s \circ \partial(\eta_2) \otimes \langle 1 \rangle \\
= 0.
\]

Hence

\[
\partial(r - s(\eta_2)) \in \text{image} \left\{ s : \bigoplus_{x \in Z} I^{n+1}(l(x)) \rightarrow \bigoplus_{x \in Z} I^{n+1}(k(x)) \right\}
\]

By our assumption, \( cd_2 l' \leq n \), so that \( I^{n+1}(l(x)) = 0 \) for \( x \in Z_l \) (see proof of (3.2)). Thus, \( \partial(r - s(\eta_2)) = 0 \) and \( r - s(\eta_2) \in I_n(Z) \), with \( e_n(r - s(\eta_2)) = x \).

**Corollary 3.4.** Suppose \( cd_2 l' \leq n \), \( Z \) satisfies \( H(n+1) \) and \( H^{n+2}(Z_l) = 0 \). Then surjectivity of \( e_{n+1} \) implies surjectivity of \( e_n \).
Proof. Surjectivity of $e_{n+1}$ implies that $(*n + 1)$ is exact. By (3.3) (B), we have, $(*n)$ exact. Since $H^1(Z, \mathcal{H}^{n+1}) \hookrightarrow H^{n+2}(Z_1) = 0$, by (A), $\phi_n = 0$ so that $e_n$ is surjective.

4. Local Fields

LEMMA 4.1. Let $k$ be a field with $cd_2 k \leq n$. Let $Z$ be a smooth integral curve over $k$, which is good. The map $e_m: I_m(Z) \to H^0(Z, \mathcal{H}^m)$ is an isomorphism for $m \geq n + 1$.

Proof. Since $cd_2 k \leq n$, we have $cd_2(k(Z)) \leq n + 1$ and $cd_2(k(x)) \leq n$ for closed points $x \in Z$. Since $Z$ is good, the maps $e_m: I^m(k(Z)) \to H^m(k(Z)), e_{m-1}: I^{m-1}(k(x)) \to H^{m-1}(k(x)), x \in Z$ are both isomorphisms for $m \geq n + 1$. (All these groups are zero for $m > n + 1$.) The commutative diagram

\[
\begin{array}{c}
0 \to I_m(Z) \to I^m(k(Z)) \to \bigoplus_{x \in Z} I^{m-1}(k(x)) \\
\downarrow e_m \downarrow \quad \downarrow \\
0 \to H^0(Z, \mathcal{H}^m) \to H^m(k(Z)) \to \bigoplus_{x \in Z} H^{m-1}(k(x))
\end{array}
\]

yields that $e_m: I_m(Z) \to H^0(Z, \mathcal{H}^m)$ is an isomorphism for $m \geq n + 1$.

LEMMA 4.2. Let $k$ and $Z$ be as in (4.1). Suppose that $Z$ is affine. Then the map $e_n: I_n(Z) \to H^0(Z, \mathcal{H}^n)$ is surjective.

Proof. Since $cd_2 k \leq n, cd_2(k(Z)) \leq n + 1$, so that $H^{n+2}(k(Z)) = 0$ and the Zariski sheaf $\mathcal{H}^{n+2} = 0$ on $Z$. Hence, $Z$ satisfies $H(n + 1)$. By (4.1), $e_{n+1}$ being surjective, $(*n + 1)$ is exact. Therefore, by (3.3) (B), $(*n)$ is exact. To show that $e_n$ is surjective, it suffices to show that $H^1(Z, \mathcal{H}^{n+1}) = 0$. The spectral sequence ([Mil], Remark 2.2.1 (b), p. 106)

\[H^p(G, H^q(Z_\bar{k})) \Rightarrow H^{p+q}(Z),\]

$G = G(\bar{k}/k), \bar{k}$ denoting the separable closure of $k$, yields, in view of $H^q(Z_\bar{k}) = 0$ for $q > 2, H^p(G, -) = 0$ for $p > n, cd_2 k$ being less than $n + 1$,

\[H^{n+2}(Z) \simeq H^n(G, H^2(Z_\bar{k})) \simeq H^n(G, \text{Pic} Z_\bar{k}/2).\]

Since $Z_\bar{k}$ is not projective, it misses at least one point of the completed curve so that Pic $Z_\bar{k}$ is divisible. Since $H^1(Z, \mathcal{H}^{n+1}) \hookrightarrow H^{n+2}(Z) = 0$, the lemma follows.

LEMMA 4.3. Let $k$ be a field with $cd_2 k \leq n$. Let $X$ be a smooth projective integral curve over $k$, which is good. Let $x_0 \in X(k)$ be a $k$-rational point. The map $e_n: I_n(X) \to H^0(X, \mathcal{H}^n)$ is surjective if and only if the restriction map $I_n(X) \to I_n(X\setminus\{x_0\})$ is surjective.

Proof. The sequence

\[H^i(k(X)) \xrightarrow{\partial} \bigoplus_{x \in X} H^{i-1}(k(x)) \xrightarrow{\text{cores}} H^{i-1}(k)\]
is a complex ([A], Satz 4.16), so that if \( x_0 \in X(k) \) and \( \alpha \in H^i(k(X)) \) is an element such that \( \partial(\alpha) \) has one possible nonzero component at \( x_0 \), then \( \alpha \in \ker \partial \). Thus, in view of [BO] the restriction map \( H^0(X, \mathcal{H}^i) \to H^0(X \setminus \{x_0\}, \mathcal{H}^i) \) is an isomorphism for every \( i \). In the following commutative diagram

\[
\begin{array}{ccc}
I_{n+1}(X) & \xrightarrow{\sim} & H^0(X, \mathcal{H}^{n+1}) \\
\downarrow & & \downarrow \delta \\
I_{n+1}(X \setminus \{x_0\}) & \xrightarrow{\sim} & H^0(X \setminus \{x_0\}, \mathcal{H}^{n+1}),
\end{array}
\]

horizontal arrows are isomorphisms, by (4.1), and the right vertical arrow is an isomorphism. Hence, the left vertical arrow is also an isomorphism. In the following commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{0} & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & I_{n+1}(X) \longrightarrow I_n(X) \longrightarrow H^0(X, \mathcal{H}^n) \\
\downarrow \delta & & \downarrow \delta \\
0 & \longrightarrow & I_{n+1}(X \setminus \{x_0\}) \longrightarrow I_n(X \setminus \{x_0\}) \longrightarrow H^0(X \setminus \{x_0\}, \mathcal{H}^n) \longrightarrow 0
\end{array}
\]

the rows are exact (4.2) and the extreme vertical arrows are isomorphisms. The lemma follows immediately.

**THEOREM 4.4.** Let \( k \) be a local field or a totally imaginary global field. For any smooth curve \( Z \) over \( k \), the map \( \epsilon_n: I_n(Z) \to H^0(Z, \mathcal{H}^n) \) is surjective for \( n \neq 2 \). The map \( \epsilon_2 \) is surjective if \( Z \) is affine. If \( Z \) is projective, and \( Z(k) \neq \emptyset \), \( \epsilon_2 \) is surjective if and only if \( Z \) has extension property for quadratic spaces: There exists \( x_0 \in Z(k) \) such that every quadratic space over \( Z \setminus \{x_0\} \) extends to \( Z \).

**Proof.** If \( k \) is a local or a global field, any smooth integral curve \( Z \) over \( k \) is good ([AEJ 2], p. 649, corollary). If \( k = \mathbb{R} \), the map \( H^0(Z, \mathcal{H}^i) \to H^0(Z, \mathcal{H}^{i+1}) \) is surjective for \( i \geq 1 \) ([W], Remark 2.4.4). The surjectivity of \( I(Z) \to H^0(Z, \mathcal{H}^n) \) implies the surjectivity of \( I_n(Z) \to H^0(Z, \mathcal{H}^n) \) for \( n \geq 1 \). If \( k = \mathbb{C} \), a non-Archimedean local field or a totally imaginary global field, we have, \( cd_zk / \leq 2 \) so that \( \epsilon_n \) is surjective for \( n \neq 2 \) by (4.1). The map \( \epsilon_2 \) is surjective if \( Z \) is not projective by (4.2). Suppose \( Z \) is projective. By (4.3), \( \epsilon_2 \) is surjective if and only if the map \( I_2(X) \to I_2(X \setminus \{x_0\}) \) is surjective. If \( X \) satisfies extension property for quadratic spaces with respect to \( x_0 \in X(k) \), then clearly \( I_2(X) \to I_2(X \setminus \{x_0\}) \) is surjective. Suppose conversely that the map \( I_2(X) \to I_2(X \setminus \{x_0\}) \) is surjective. Let \( q \) be a quadratic space on \( X \setminus \{x_0\} \). To show that \( q \) extends to \( X \), we need to check that the second residue \( \partial_{x_0}(q) = 0 \). If rank \( q \) is odd, we may replace \( q \) by \( q \perp \langle 1 \rangle \) and assume that rank \( q \) is even. We have \( \text{disc} q \in H^1(X \setminus \{x_0\}, \mu_2) = H^1(X, \mu_2) \) so that \( \text{disc} q \) is integral on \( X \). Replacing \( q \) by
$q \perp \langle 1, \text{disc } q \rangle$, we may assume that $\text{disc } q$ is trivial. Thus, $[q] \in I_2(X \setminus \{x_0\}) = I_2(X)$. Therefore $\tilde{e}_{x_0}(q) = 0$ and $q$ extends to $X$.

**COROLLARY 4.5.** Let $Z$ be a smooth projective integral curve over $k$, which is a local field or a totally imaginary number field. Suppose $\Omega_Z$ is a square in $\text{Pic } Z$ and $Z(k) \neq \phi$. Then $\tilde{e}_n$ is an isomorphism for all $n$.

5. Surjectivity of $e_4$ for Curves over Number Fields:
   **Mod-2 Signatures**

Let $k$ be a number field and $Z$ a smooth curve over $k$. Let $H^*_4(k(Z))$ denote the $'-1$ torsion’ subgroup of $H^*_4(k(Z))$: $H^*_4(k(Z)) = \{ \xi \in H^*_4(k(Z), \xi \cup (\chi_{-1}) = 0 \text{ for some positive integer } r \}$. Let $h_n: H^*_n(k(Z)) \to \mathcal{C}(P_{k(Z)}, \mathbb{Z}/2)$ be the map defined in [AEJ 1], ($P_{k(Z)}$ denoting the space of orderings of $k(Z)$). Then $\ker h_n = H^*_n(k(Z))$, in view of the results of ([AEJ 1], Lemma 2.2). We recall the mod-2 signature homomorphism

$$h_n: H^0(Z, \mathcal{H}^n) \to \mathcal{C}(\text{Spec } Z, \mathbb{Z}/2)$$

defined in [CTPa]. If $(x, v) \in \text{Spec } Z, x \in \text{Spec } Z, v: k(x) \to L$ an injection into a real closed field $L$, and $\alpha \in H^0(Z, \mathcal{H}^n)$, then $h_n(\alpha)(x, v) = H^n(v)(\tilde{\alpha}_x), \tilde{\alpha}_x$ denoting the restriction of $\alpha_x$ to the fibre at $x, x \in H^0(\mathcal{O}_{Z, x})$ being a lift of $\alpha$; we identify $H^*(L) \simeq \mathbb{Z}/2$ for a real closed field $L$. The diagram

$$
\begin{array}{ccc}
H^0(Z, \mathcal{H}^n) & \xrightarrow{h_n} & \mathcal{C}(\text{Spec } Z, \mathbb{Z}/2) \\
\downarrow & & \downarrow \text{res} \\
H^*(k(Z)) & \xrightarrow{h_n} & \mathcal{C}(P_{k(Z)}, \mathbb{Z}/2)
\end{array}
$$

is commutative, where $P_{k(Z)}$ is the set of generic orderings of $Z$, treated as a subspace of $\text{Spec } Z$. Further, the total signature homomorphism $\text{sgn}: W(Z) \to \mathcal{C}(\text{Spec } Z, \mathbb{Z})$ maps $I_n(Z)$ into $2^n\mathbb{Z}$ and we have a commutative diagram

$$
\begin{array}{ccc}
I_n(Z) & \xrightarrow{\text{sgn}} & \mathcal{C}(\text{Spec } Z, 2^n\mathbb{Z}) \\
\downarrow e_n & & \downarrow \\
H^0(Z, \mathcal{H}^n) & \xrightarrow{h_n} & \mathcal{C}(\text{Spec } Z, 2^n\mathbb{Z}/2^{n+1}\mathbb{Z}) = \mathcal{C}(\text{Spec } Z, \mathbb{Z}/2).
\end{array}
$$

**LEMMA 5.1.** $H^*_4(k(Z)) = 0$.

**Proof.** Suppose first that $k$ is totally imaginary. Since $cd_z k \leq 2, cd_2 k(Z) \leq 3$, so that $H^4(k(Z)) = 0$. Suppose $k$ is not totally imaginary. Let $l = k(\sqrt{-1})$. Then $H^*(l(Z)) = 0$ for $n \geq 4$ and the exact sequence

$$0 = H^4(l(Z)) \to H^4(k(Z)) \xrightarrow{\cup\chi_1} H^5(k(Z)) \to H^5(l(Z)) = 0$$

yields that $\cup\chi_1$ is an isomorphism. Thus, $H^*_4(k(Z)) = 0$. 
LEMMA 5.2. Let \( Y \) be a smooth affine curve over a number field \( k \) and let \( \text{Spec}, Y \) denote the real spectrum of \( Y \). If \( \text{Spec}, Y = U_1 \cup U_2 \) is decomposition into disjoint open sets, then there exists a quadratic space \( q \) in \( I_4(Y) \) with \( \text{sgn} q = 2^4 \) on \( U_1 \) and \( \text{sgn} q = 0 \) on \( U_2 \).

**Proof.** By a theorem of Mahé [M], there exists an integer \( N \geq 1 \), and a quadratic space \( q \) on \( Y \) such that \( \text{sgn} q = 2N \) on \( U_1 \), \( \text{sgn} q = 0 \) on \( U_2 \), \( \text{sgn}: W(Y) \to \mathcal{C}(\text{Spec}, Y, \mathbb{Z}) \) being the total signature homomorphism. Replacing \( q \) by \( \langle 1 \rangle \otimes q \), we may assume that the class of \( q \) in \( W(Y) \) belongs to \( I_4(Y) \). Suppose \( N > 5 \). In view of the commutative diagram

\[
\begin{array}{ccc}
I_4(Y) & \overset{\text{sgn}}{\longrightarrow} & \mathcal{C}(\text{Spec}, Y, 2^4 \mathbb{Z}) \\
\downarrow e_4 & & \downarrow \\
H^0(Z, \mathcal{H}^4) & \overset{h}{\longrightarrow} & \mathcal{C}(\text{Spec}, Y, 2^4 \mathbb{Z}/2^5 \mathbb{Z}) = \mathcal{C}(\text{Spec}, Y, \mathbb{Z}/2),
\end{array}
\]

since \( N \geq 5 \), \( \text{sgn} q \equiv 0 \pmod{2^5} \) so that \( h_4 e_4(q) = 0 \). By [AEJ 1], \( e_4(q) \in H^4_4(k(Y)) = 0 \) by (5.1). Thus, \( q \in I_4(Z) \). We have a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & I_4(Y) \longrightarrow I^4(k(Y)) \longrightarrow \bigoplus_{x \in Y} I^3(k(x)) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & I_5(Y) \longrightarrow I^5(k(Y)) \longrightarrow \bigoplus_{x \in Y} I^4(k(x))
\end{array}
\]

The two right vertical maps are isomorphisms since by (1.5), \( I^{n-1}(l(Y)) = 0 \), \( I^{n-1}(l(x)) = 0 \) for \( x \in Y \), for \( n \geq 4 \). Hence, \( \otimes \langle 1 \rangle : I_4(Y) \to I_5(Y) \) is an isomorphism. Thus, there exists \( q_1 \in I_4(Y) \) with \( q_1 \otimes \langle 1 \rangle = q \), \( \text{sgn} q_1 = 2^{N-1} \) on \( U_1 \) and \( \text{sgn} q_1 = 0 \) on \( U_2 \). Repeating this process, one obtains \( q \in I_4(Y) \) with \( \text{sgn} q = 2^4 \) on \( U_1 \) and \( \text{sgn} q = 0 \) on \( U_2 \).

LEMMA 5.3. Let \( Z/k \) be a smooth affine curve over a number field \( k \). Then the map \( e_4: I_4(Z) \to H^0(Z, \mathcal{H}^4) \) is surjective.

**Proof.** Let \( h: H^0(Z, \mathcal{H}^4) \to \mathcal{C}(\text{Spec}, Z, \mathbb{Z}/2) \) be the mod-2 signature homomorphism. The kernel of \( h: H^4(k(Z)) \to \mathcal{C}(P_{k(Z)}, \mathbb{Z}/2) \) is precisely \( H^4_4(k(Z)) \) which is zero by (5.1). Therefore the map \( h: H^0(Z, \mathcal{H}^4) \to \mathcal{C}(\text{Spec}, Z, \mathbb{Z}/2) \) is injective. Let \( x \in H^0(Z, \mathcal{H}^4) \). Let \( U_1 \cup U_2 = \text{Spec}, Z \) be a decomposition into disjoint open sets with \( h(x) = 1 \) on \( U_1 \) and \( h(x) = 0 \) on \( U_2 \). By (5.2) there exists \( q \in I_4(Y) \) such that \( \text{sgn} q = 2^4 \) on \( U_1 \) and \( \text{sgn} q = 0 \) on \( U_2 \). This implies that \( h \circ e_4(q) = h(x) \). Since \( h \) is injective, we have \( e_4(q) = x \).

PROPOSITION 5.4. Let \( Z \) be any smooth curve over a number field \( k \). Then the map \( e_4: I_4(Z) \to H^0(Z, \mathcal{H}^4) \) is surjective.
Proof. Let $Y = \text{Spec} \, A \subset Z$ be an affine open subscheme such that $Y_{k_i}(k_i) = X_{k_i}(k_i)$ for every real completion $k_i$ of $k$. We have the following commutative diagram

$$
\begin{array}{ccc}
I_4(X) & \xrightarrow{\mathcal{E}_4} & H^0(X, \mathcal{H}^4) \\
\downarrow & & \downarrow \\
I_4(Y) & \xrightarrow{\mathcal{E}_4} & H^0(Y, \mathcal{H}^4)
\end{array}
$$

with the vertical maps injective and the lower horizontal map surjective, by (5.3). The surjectivity of $\mathcal{E}_4$ would follow provided we show that $I_4(X) = I_4(Y)$.

Let $\beta \in I_4(Y)$. For $x \in Y$, $\partial_x(\beta) = 0$. For $x \in X \setminus Y$, $k(x)$ is totally imaginary so that $I^3(k(x)) = 0$ and hence $\partial_x(\beta) = 0$. Thus $\partial(\beta) = 0$ and $\beta \in I_4(X)$. Thus, it suffices to exhibit such a $Y \subset X$. Let $X \hookrightarrow \mathbb{P}^n$ be an embedding of $X$ in a projective space. Let $\mathbb{P}^n_k \hookrightarrow \mathbb{P}^n$ be the Veronese embedding for the hypersurface $\Sigma_{0 \leq i \leq n} x_i^2$ in $\mathbb{P}^n_k$, $0 \leq i \leq n$, denoting the coordinate planes in $\mathbb{P}^n_k$. Let $H \subset \mathbb{P}^n_k$ the hyperplane with $H \cap \mathbb{P}^n_k = \{ \Sigma x_i^2 = 0 \}$. Then $Y = X \setminus H$ is affine and has the required properties.

6. Main Theorem

We come to the main result of this paper.

**Theorem 6.1.** Let $Z$ be a smooth quasi-projective integral curve over a global field $k$ of characteristic not 2. Then, the graded Witt ring of $Z$ is isomorphic to the graded (mod-2) cohomology ring of $Z$ in the following cases.

1. $Z$ is affine.
2. $Z$ is projective, $Z$ has a $k$-rational point and the canonical line bundle of $Z$ is a square in $\text{Pic} \, Z$.

In fact, we prove the following more precise

**Theorem 6.2.** Let $Z$ be a smooth quasi-projective curve over a global field $k$ of characteristic not 2. Then the maps $e_n : I_n(Z) \to H^0(Z, \mathcal{H}^n)$ are surjective for $n \neq 2$. The map $e_2$ is surjective if $Z$ satisfies one of the two conditions (1) and (2) of (6.1).

In view of (4.4) and (4.5) it is enough to prove the theorem in the case where $k$ is a number field with at least one real completion. We set $l = k(\sqrt{-1})$. We denote by $Z$ a smooth quasi-projective curve over $k$.

**Lemma 6.3.** Let $Z$ be as in (6.2). Then the map $e_n : I_n(Z) \to H^0(Z, \mathcal{H}^n)$ is surjective for $n \geq 4$.

**Proof.** Since $cd_2(l(Z)) \leq 3, I^n(l(Z)) = 0, I^{n-1}(l(x)) = 0$ for closed points $x \in Z$, for $n \geq 4$. In the following commutative diagram, for $n \geq 4$, the rows are exact and the...
two right vertical maps are isomorphisms

\[
\begin{array}{ccc}
0 & \longrightarrow & I_n(Z) \\
\downarrow & & \downarrow \otimes \langle 1 \rangle \\
0 & \longrightarrow & I_n+1(Z)
\end{array}
\quad \begin{array}{ccc}
I^n(k(Z)) & \longrightarrow & \bigoplus_{x \in Z} I^{n-1}(k(x)) \\
\downarrow & & \downarrow \otimes \langle 1 \rangle \\
I^{n+1}(k(Z)) & \longrightarrow & \bigoplus_{x \in Z} I^n(k(x))
\end{array}
\]

Hence, the left vertical map is an isomorphism. Since \( H^n(l(Z)) = 0, H^{n-1}(l(x)) = 0 \) for \( n \geq 4, x \in Z \), a similar diagram as above with \( I \) replaced by \( \mathcal{H} \) yields that the map \( \cup_{\chi^{-1}}: H^0(Z, \mathcal{H}^n) \rightarrow H^0(Z, \mathcal{H}^{n+1}) \) is an isomorphism for \( n \geq 4 \). The commutative diagram

\[
\begin{array}{ccc}
I_n(Z) & \xrightarrow{e_n} & H^0(Z, \mathcal{H}^n) \\
\downarrow \otimes \langle 1 \rangle & & \downarrow \cup_{\chi^{-1}} \\
I_{n+1}(Z) & \xrightarrow{e_{n+1}} & H^0(Z, \mathcal{H}^{n+1})
\end{array}
\]

for \( n \geq 4 \) shows that the surjectivity of \( e_4 \) implies the surjectivity of \( e_n \) for \( n \geq 4 \). The surjectivity of \( e_4 \) is proved in (5.4).

**Lemma 6.4.** For \( Z \) as in (6.2), the map \( e_3: I_3(Z) \rightarrow H^0(Z, \mathcal{H}^3) \) is surjective.

**Proof.** We have \( cd_2 l \leq 2 \). Since the map \( \cup_{\chi^{-1}}: H^0(Z, \mathcal{H}^n) \rightarrow H^0(Z, \mathcal{H}^{n+1}) \) is an isomorphism for \( n \geq 4, Z \) satisfies \( H(4) \). Further, \( e_4 \) being surjective (5.4), (4.4) is exact. Thus, to show that \( e_3 \) is surjective, it is enough by (3.3) to show that \( H^1(Z, \mathcal{H}^4) = 0 \). We have,

\[
H^1(Z, \mathcal{H}^4) \hookrightarrow H^5(Z_1) = 0,
\]

since \( cd_2 l \leq 2 \) and \( cd_2 Z_1 \leq 2 \), and in view of the convergence of the spectral sequence

\[
H^p(G, H^q(Z_1)) \Rightarrow H^n(Z_1).
\]

**Lemma 6.5.** For \( Z \) as in (6.2), the curve \( Z \) satisfies \( H(3) \); i.e. \( \cup_{\chi^{-1}}: H^0(Z, \mathcal{H}^3) \rightarrow H^0(Z, \mathcal{H}^4) \) is surjective.

**Proof.** Let \( x \in H^0(Z, \mathcal{H}^4) \) and let \( x_i \) denote its image in \( H^0(Z_{k_i}, \mathcal{H}^4) \), where \( \{k_i, 1 \leq i \leq r\} \) denote the set of real completions of \( k \). Since \( k_i \) is real closed, \( l_i = k_i(\sqrt{-1}) \) is algebraically closed so that \( H^1(l_i(Z)) = 0, H^{-1}(l_i(x)) = 0 \) for \( x \in Z_{l_i}, j \geq 2 \). Thus the commutative diagram

\[
\begin{array}{ccc}
H^3(k_i(Z)) & \xrightarrow{\cup_{\chi^{-1}}} & H^4(k_i(Z)) \\
\downarrow \otimes \langle 1 \rangle & & \downarrow \otimes \langle 1 \rangle \\
\bigoplus_{x \in Z_{l_i}} H^2(k_i(x)) & \xrightarrow{\cup_{\chi^{-1}}} & \bigoplus_{x \in Z_{l_i}} H^3(k_i(x))
\end{array}
\]
yields an isomorphism

$$\bigcup_{\chi < 1} H^0(Z_{k_i}, \mathcal{H}^3) \cong H^0(Z_{k_0}, \mathcal{H}^4).$$

Let $\beta_i \in H^0(Z_{k_i}, \mathcal{H}^3)$ be such that $\beta_i \cup_{\chi < 1} = \alpha_i$. In view of [Ka]\S4, there exists $\beta \in H^0(Z, \mathcal{H}^3)$ with image $\beta_i \in H^0(Z_{k_i}, \mathcal{H}^3)$. Replacing $\alpha$ by $\alpha - \beta \cup_{\chi < 1}$, we may assume that $\alpha$ maps to zero in $H^0(X_{k_i}, \mathcal{H}^4)$, $1 \leq i \leq r$. This would imply that $\alpha = 0$ in view of the following

**Lemma 6.6.** For $Z$ as in (6.2), the map

$$H^0(Z, \mathcal{H}^4) \to \bigoplus_{1 \leq i \leq r} H^0(Z_{k_i}, \mathcal{H}^4)$$

is injective.

**Proof.** Let $\text{Spec } A = Y \subset Z$ be chosen as in the proof of (5.4). Let $\alpha \in H^0(Z, \mathcal{H}^4)$ map to zero in $H^0(Z_{k_i}, \mathcal{H}^4)$, $1 \leq i \leq r$. Let $\text{Spec } Y = U_1 \cup U_2$ be such that $h(\alpha)|U_1 = 1, h(\alpha)|U_2 = 0$. As in (5.4), we may choose $q \in I_4(Y)$ such that $\text{sgn } q = 2^s$ on $U_1$, $\text{sgn } q = 0$ on $U_2$ and $e_4(q) = \alpha$. We have, $q \otimes k_i \in I_4(Y_{k_i})$ mapping to zero in $H^0(Y_{k_i}, \mathcal{H}^4)$ and, hence, $q \otimes k_1 \in I_5(Y_{k_1})$ and the signature of $q \otimes k_i$ is divisible by $2^s$. The projection $\pi_i: Y_{k_i} \to Y$ induces a map $\text{Spec } Y_{k_i} \to \text{Spec } Y$ and a map $\mathcal{O}(\text{Spec } Y, Z) \to \mathcal{O}(\text{Spec } Y_{k_i}, Z)$ with $\pi_i(\text{sgn } q) = \text{sgn } (q \otimes k_i)$. Since $\text{sgn } q$ is not divisible by $2^s$, it follows that $\text{sgn } (q \otimes k_i) = 0$. Since $I^2(k_i(Z))$ is torsion free, $q \otimes k_i = 0, 1 \leq i \leq r$. Since the map $I^4(k(Z)) \to \bigoplus_{1 \leq i \leq r} I^4(k_i(Z))$ is injective, $[q] = 0$ in $W(k(Z))$ and in particular, $e_4(q) = \alpha = 0$.

**Lemma 6.7.** For $Z$ as in (6.2), the map $e_2$ is surjective if either $Z$ is affine or if $Z$ is projective and the canonical line bundle of $Z$ is a square in $\text{Pic } Z$.

**Proof.** We have $cd_2 \leq 2$ and by (6.5), $Z$ satisfies $H(3)$. Since $e_2$ is surjective (6.4), to prove the surjectivity of $e_2$ it is enough to show by (3.3) that $\varphi_2$ is zero. Suppose $Z$ is not projective. Since $cd_2 \leq 2$ and $H^2(Z_t) = \text{Pic } Z_t/2 = 0$, $Z_t$ being not projective, it follows that $H^3(Z_t) = 0$ and $H^1(Z_t, \mathcal{H}^3)$ which is a subgroup of $H^4(Z_t)$ is also zero. Thus $\varphi_2 = 0$ in this case. Suppose $Z$ is projective and the canonical line bundle of $Z$ is a square in $\text{Pic } Z$. Let $\alpha \in H^0(Z, \mathcal{H}^2)$. The map $\varphi_2$ commutes with base change and for each completion $k_v$ of $k$, $\varphi_2$ is surjective for $Z_{k_v}$, under the given hypothesis (4.4). Thus $\varphi_2(\alpha) \in H^1(Z, \mathcal{H}^3) \hookrightarrow H^4(Z)$ maps to zero in $H^4(Z_{k_v})$ for each $v$. By a result of Kato [Ka], the map $H^4(Z) \to \bigoplus_{1 \leq i \leq r} H^4(Z_{k_i})$ is injective, so that $\varphi_2(\alpha) = 0$.

This completes the proof of the lemma and the proof of Theorem 6.1.

**Remark.** The condition (2) on $Z$ in (6.1) can be replaced by the condition that $Z(k_v) \neq \phi$ and the canonical line bundle on $Z_{k_v}$ is a square in $\text{Pic } Z_{k_v}$ for each completion $k_v$ of $k$.

**Corollary 6.8.** Let $X$ be a smooth projective curve over a global field $k$ of characteristic different from 2, with a $k$-rational point. Suppose $2\text{Pic } X_k = 2\text{Pic } X$. Then the graded Witt ring of $X$ is isomorphic to the graded cohomology ring of $X$.

**Proof.** Immediate from (6.1) and (2.1).
COROLLARY 6.9. Let \( k \) be a global field of characteristic different from 2 and let \( \pi: X \to \mathbb{P}^1 \) be a double covering with \( X \) smooth (\( X \) hyperelliptic). Then if \( X(k) \neq \emptyset \) and genus \( X \leq 1 \), or if genus \( X \geq 2 \) and \( \pi \) has a rational ramification point, the graded Witt ring of \( X \) is isomorphic to the graded cohomology ring of \( X \).

Proof. If genus \( X = 0 \), and \( X(k) \neq \emptyset \), \( X \cong \mathbb{P}^1 \), \( \Omega_X = \mathcal{O}(-2) \). If genus \( X = 1 \), \( \Omega_X \cong \mathcal{O}_X \). If genus \( X \geq 2 \) and \( x \in \mathbb{P}^1 \) is a \( k \)-rational point of ramification for \( \pi \) and if \( \pi(P) = x \), then \( \Omega_X = \mathcal{O}(2(g - 1)P) \). Thus, in all these cases, \( \Omega_X \) is a square in \( \text{Pic} X \).

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References


