Disjoint Stable Matchings

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Outline

- Background
  - Stable Matchings
  - Gale-Shapley Algorithm
  - Extended Gale-Shapley Algorithm
  - The Lattice Structure
- Disjoint Stable Matchings
  - Disjoint Perfect Matchings
  - Disjoint Stable Matchings
  - Algorithm to find Disjoint Stable Matchings
  - Correctness and Running time of the Algorithm
- Rotations
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  - Properties of Rotational poset
  - Analogous Algorithm

Marriage Matching Instance

A marriage matching instance of *size* n involves two disjoint sets of size n, the men and the women. Associated with each person is a *strictly* ordered *preference list* containing *all* the members of the opposite sex. Person p prefers q to r, where q and r are of the opposite sex of p, if and only if q precedes r on p's preference list.

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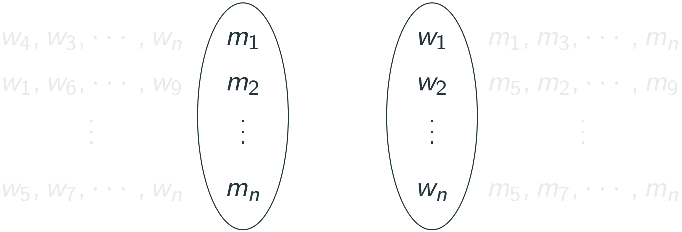

Set of men   Set of women

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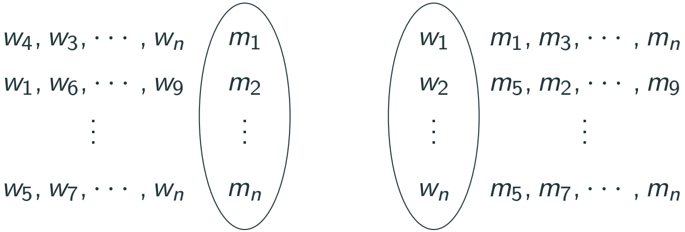
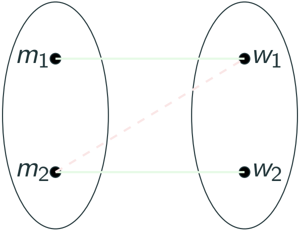


Set of men Set of women

If (m, w) are matched in a matching M, we say $m = p_M(w)$ and $w = p_M(m)$

Blocking Pair

A man m and a woman w are said to *block* a matching M, or the pair (m, w) is said to be a *blocking pair* for M, if m and w are not partners in M, but m prefers w to $p_M(m)$ and w prefers m to $p_M(w)$. A matching with no blocking pair is called a *stable* matching, and is otherwise *unstable*.



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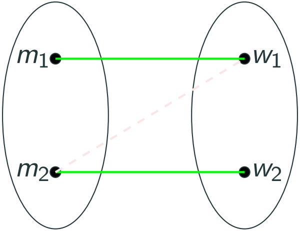

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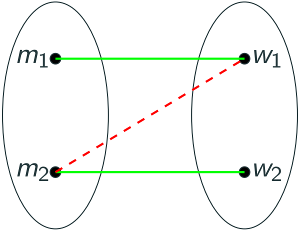

Stable Matchings

Stable Matching

A matching with no blocking pair

Checking stability: $O(n^2)$

Stable and Fixed Pairs

Stable Pair

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Fixed Pair

A pair (m, w) is called as a *fixed pair* if m and w are partners in at all stable matchings.

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Gale-Shapley Algorithm

Algorithm 1 Gale-Shapley

```
procedure FIND STABLE MATCHING(M)
    assign each person to be free
    while some man m is free do
        w ← first woman on m's list to whom m hasn't proposed
        if w is free then
            assign m and w to be engaged to each other
        else
            if w prefers m to her current matched partner m′ then
                assign m and w to be engaged and m′ to be free
            else
                w rejects m                                ▷ m remains free
            end if
        end if
    end while
    return Stable matching consisting of n engaged pairs
end procedure
```

Gale-Shapley Algorithm

m_1 :	w_2	w_3	w_1
m_2 :	w_3	w_1	w_2
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Men's Preference

w_1 :	m_1	m_2	m_3
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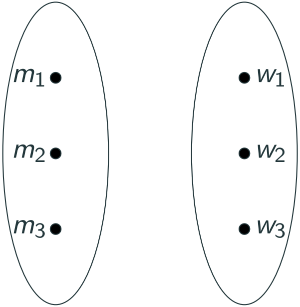
Women's Preference



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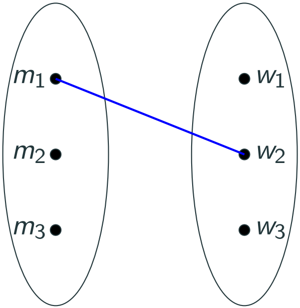

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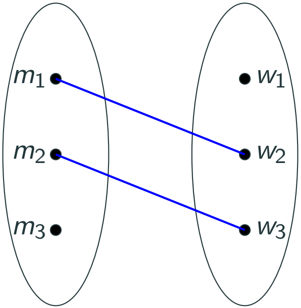

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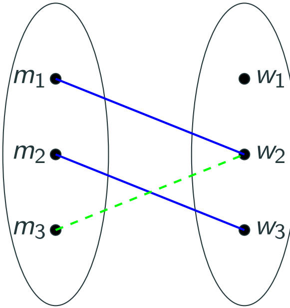

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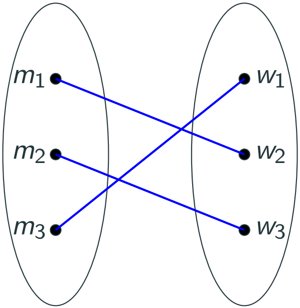

Gale-Shapley Algorithm - Key Results

1. Every marriage instance has a stable matching.
2. All possible execution of the Gale-Shapley algorithm yields the same result.
3. It results in *"Man-optimal"* stable matching.

   Man-optimal: Every man is matched with his most favored partner among all stable partners.
4. Reversing roles, i.e, women proposing, results in *"Woman-optimal"* stable matching.

   Woman-optimal: Every woman is matched with her most favored partner among all stable partners.
5. The man-optimal stable matching is *woman-pessimal*, and vice-versa.

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Extended Gale-Shapley Algorithm

Algorithm 2 Extended Gale-Shapley

```
procedure GS-EXTENDED(M)                              ▷ M is an SM instance
    assign each person to be free
    while some man m is free do
        w ← first woman on m's list
        if some man p is engaged to w then
            assign p to be free
        end if
        assign m and w to be engaged to each other
        for each successor m' of m on w's list do
            delete w on m''s list
            delete m' on w's list                     ▷ deleting the pair (m', w)
        end for
    end while
    return Stable matching consisting of n engaged pairs
end procedure
```

Run of Extended GS Algorithm

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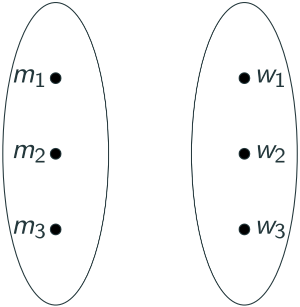

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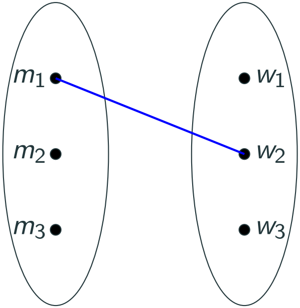

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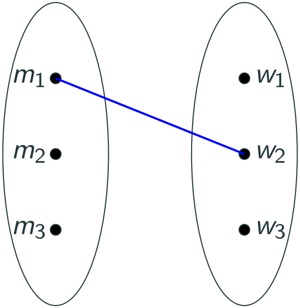

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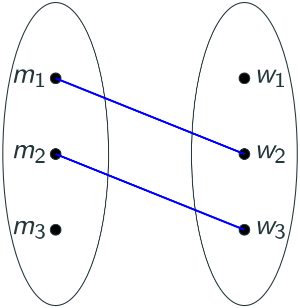

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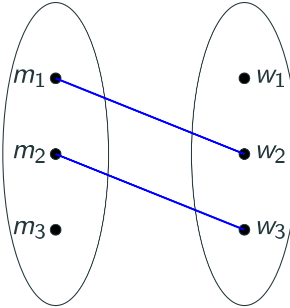

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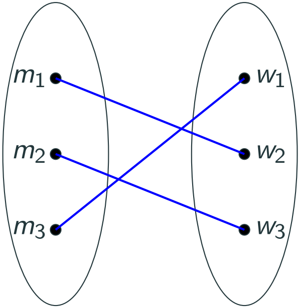

GS-lists

MGS-list

Final preference lists generated by the extended Gale-Shapley algorithm with men as proposers are called as *man-oriented Gale-Shapley lists* or *MGS-lists*.

WGS-list

Final preference lists generated by the extended Gale-Shapley algorithm with *women* as proposers are called as *woman-oriented Gale-Shapley lists* or *WGS-lists*.

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GS-lists

GS-list

Intersection of MGS-list and WGS-list.

Note: GS-lists can be obtained by applying man-oriented extended Gale-Shapley algorithm to get MGS-lists and then, starting with the MGS-lists, applying woman-oriented extended GS algorithm.

Extended GS Algorithm Key Results

1. all stable matchings are contained in the GS-lists.
2. no matching (stable or otherwise) contained in the GS-lists can be blocked by a pair that is not in the GS-lists.
3. In the man-optimal (respectively woman-optimal) stable matching, each man is partnered by the first (respectively last) woman on his GS-list, and each woman by the last (respectively first) man on hers.

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The Lattice Structure

A person x is said to *prefer* a matching M to a matching M' if x prefers $p_M(x)$ to $p_{M'}(x)$.

Domination

A stable matching M is said to *dominate* a stable matching M', written $M \preceq M'$, if every man has at least as good a partner in M as he has in M'.i.e., every man either prefers M to M' or is indifferent between them. M *disjointly dominates* $M'(M \prec M')$ if $M \preceq M'$ and $M \cap M' = \varnothing$.

m_1 :				
m_2 :				
m_3 :				
m_4 :				

Men's Preference

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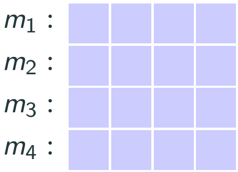
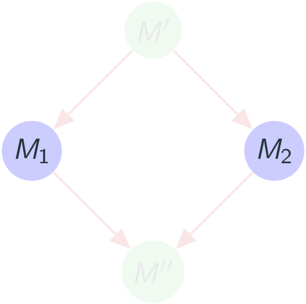
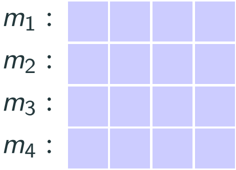


Men's Preference

Meet and Join





Men's Preference

$$M' = \{(m, w) \mid w = best(p_{M_1}(m), p_{M_2}(m))\}$$

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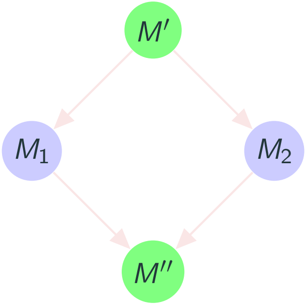

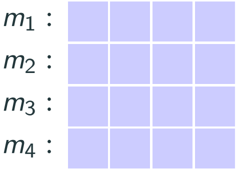

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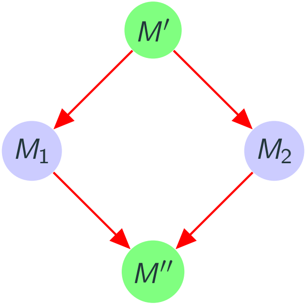

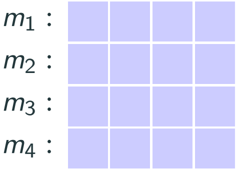

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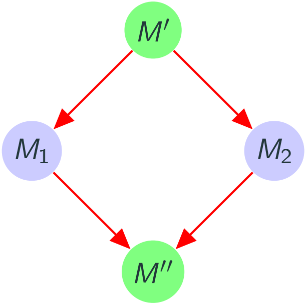

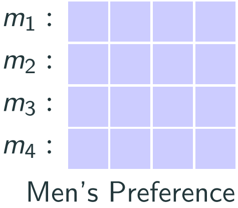

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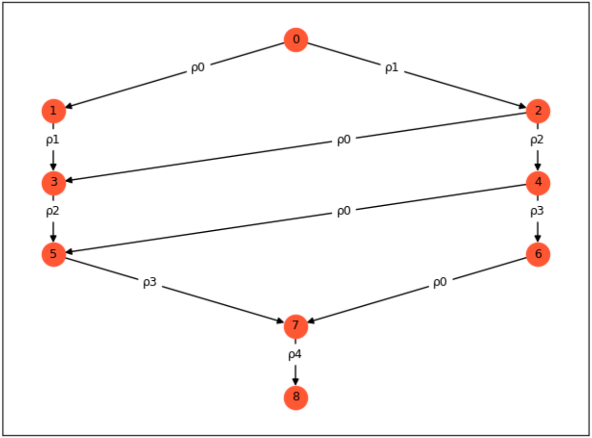
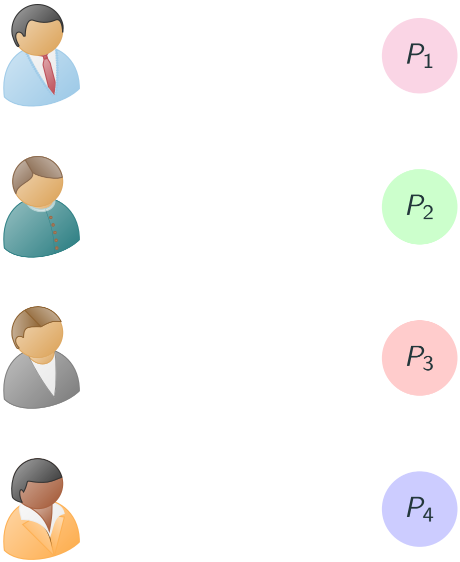
The Lattice Structure

Set of all stable matchings form a distributive lattice under the *Domination* domination.



Disjoint Stable Matchings

Why do we need them?



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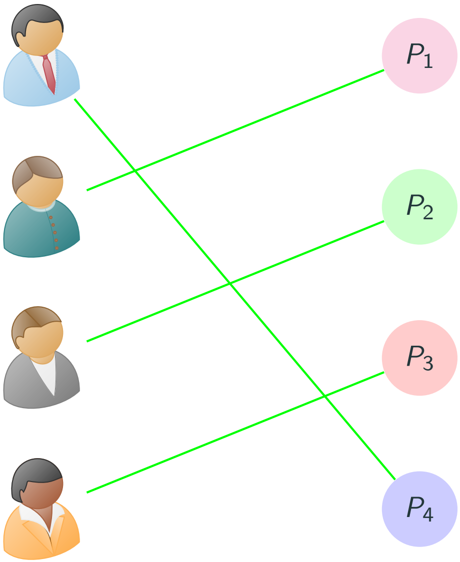

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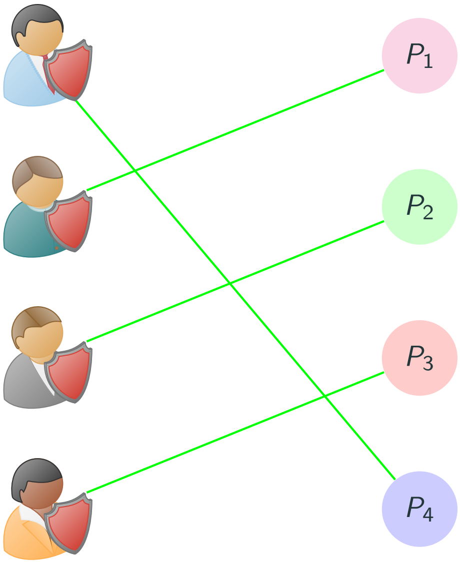

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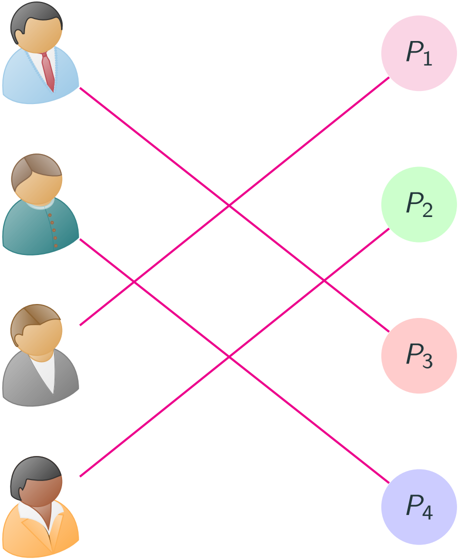

Problem Statement

For a given marriage instance, find a largest set S of disjoint stable matchings.

Existence of Disjoint Stable Matchings

Does there exist a marriage matching instances with disjoint stable matchings?

$m_1 : w_1, w_2, w_3$	$w_1 : m_2, m_3, m_1$
$m_2 : w_2, w_3, w_1$	$w_2 : m_3, m_1, m_2$
$m_3 : w_3, w_1, w_2$	$w_3 : m_1, m_2, m_3$

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$$
\begin{array}{ll}
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\end{array}
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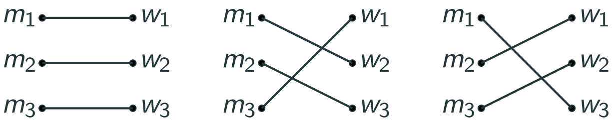


Necessary Condition

If the man-optimal and the woman-optimal stable matchings share a common edge (m, w), then (m, w) is in every stable matching.

This is because w is both the best stable partner and the worst stable partner of m.

So, to have disjoint stable matchings, man-optimal and woman-optimal matchings must be disjoint.

Algorithm: Disjoint Stable Matchings

Algorithm 3 Disjoint Stable Matchings

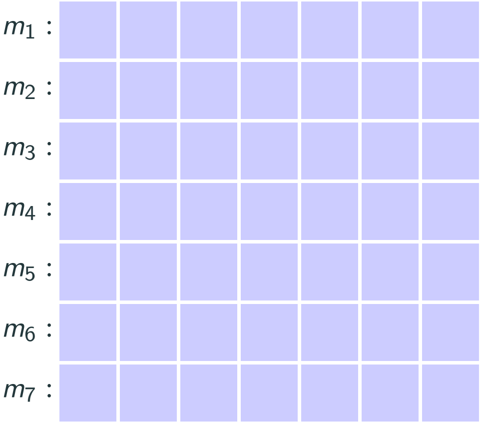
procedure FIND MAXIMUM SET OF DISJOINT STABLE MATCHINGS(M)
    $S \leftarrow \varnothing$ ▷ S: Set of disjoint matchings. Initialize S to be an empty set
    $M' \leftarrow M$.ReverseRoles ▷ Men renamed as women and women as men
    $M_z \leftarrow$FINDSTABLEMATCHING(M') ▷ GS Algorithm: Woman-optimal
    $X \leftarrow$GS-EXTENDED(M) ▷ calling Algorithm 2 modifies M's list
    while $X \cap M_z = \varnothing$ **do**
        $S \leftarrow S \cup \{X\}$
        for every man m **do**
            Delete first woman w on m's list ▷ First woman is $p_X(m)$
            Delete last man on w's list ▷ Last man is $p_X(w)$
            Mark m as *free*
        end for
        $X \leftarrow$GS-EXTENDED(M) ▷ Get a new disjoint matching as X
    end while
    $S \leftarrow S \cup \{M_z\}$
    return S
end procedure

Termination and Time Complexity

In every iteration, we delete at least one entry from the preference list. As the size of preference list is $2n^2$, the algorithm **terminates**.

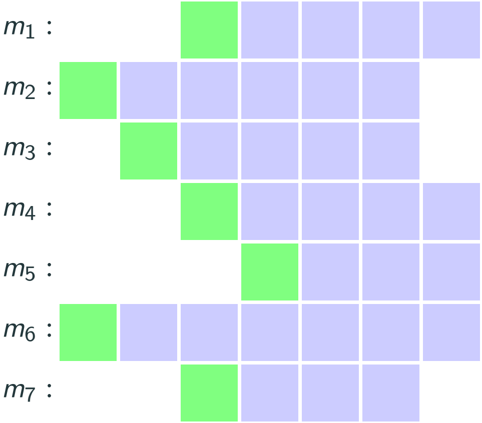
For the same reason, the running time of the algorithm is $\mathrm{O}(\mathrm{n}^2)$.

Run of Disjoint GS Algorithm



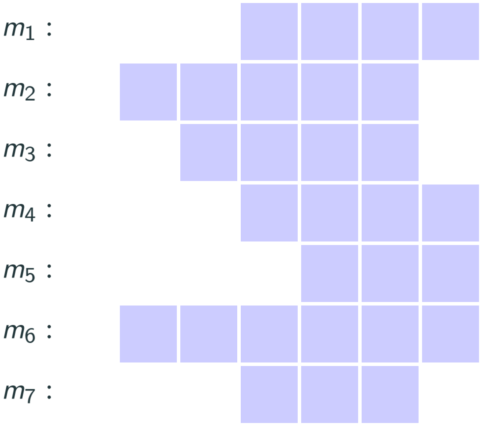
Men's preference list

Run of Disjoint GS Algorithm



Men's Preference

Run of Disjoint GS Algorithm



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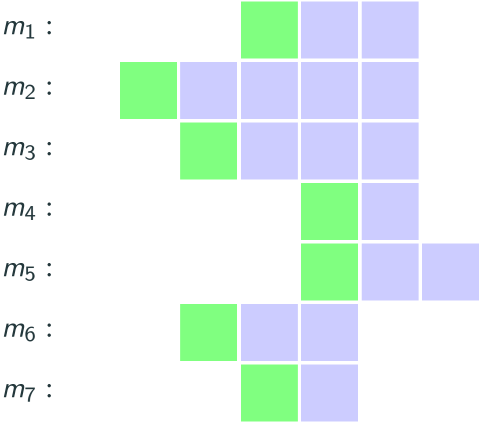

Men's Preference

Disjoint Stable Matchings

Lemma 1

Each M_i in the set $S = \{M_0, M_1, \cdots, M_n = M_z\}$ is a perfect matching.

Note: It does not come freely from Extended GS!

It only guarantees one-one.

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Disjoint Stable Matchings

Lemma 2

All the matchings in the set S are stable matchings.

Lemma 3

If $M_0, M_1, \cdots, M_n = M_z$ are the matchings discovered by the algorithm 3 in this order, then $M_0 \prec M_1 \prec \cdots \prec M_n = M_z$.

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Disjoint Stable Matchings

Lemma 4

In any arbitrary execution E of the algorithm 3, for any man m, $p_{M_i}(m)$ is the best stable partner of m when, for every *man, stable partners from $M_0, M_1, \cdots, M_{i-1}$ are disallowed.*

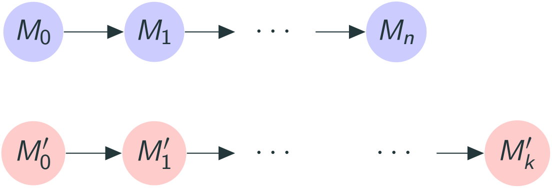
m_1 :							
m_2 :							
m_3 :							
m_4 :							
m_5 :							
m_6 :							

Longest Chain of Disjoint Stable matchings

Lemma 5

The algorithm 3 gives the longest chain of disjoint stable matchings.

Proof:



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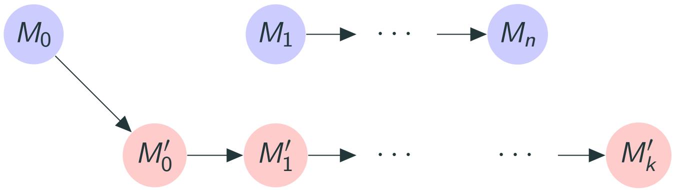

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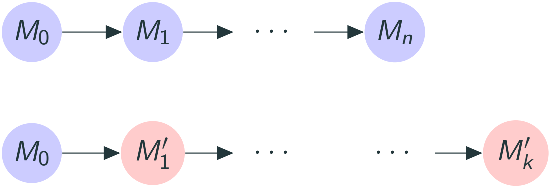

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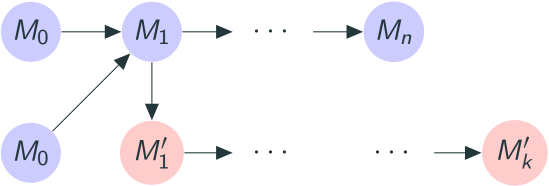

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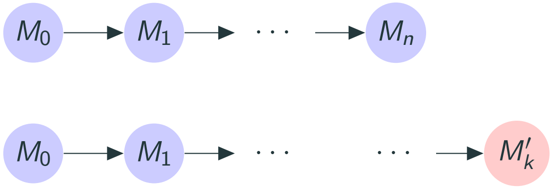

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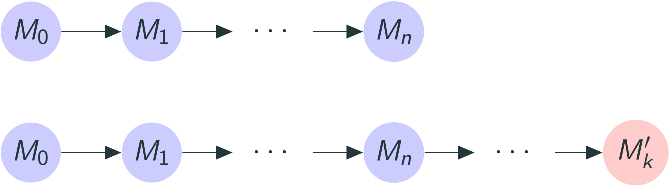

Disjoint Stable Matching

Theorem 6 (Teo, C.-P. and Sethuraman, J. (1998))

Let I be an SM instance and let $\mathcal{T}$ be a set of stable matchings in I. Let $\alpha_{j,\mathcal{T}}$ (respectively $\beta_{j,\mathcal{T}}$) denote the set of pairs obtained by assigning each man m_i (woman w_i) to $p_{j,\mathcal{T}}(m_i)$ $(p_{j,\mathcal{T}}(w_i))$, the jth element in the sorted multiset $P_{\mathcal{T}}(m_i) = \{w_i | (m_i, w_i) \in M, M \in \alpha_{j,\mathcal{T}}\}$ (respectively $P_{\mathcal{T}}(w_i)$). Then, each of $\alpha_{j,\mathcal{T}}$ and $\beta_{j,\mathcal{T}}$ is a stable matching.

Given stable matchings $M_1, M_2, \cdots, M_k$,

$$M'_i = \{(m, w) \,|\, w \text{ is the i-th women in the } \textbf{sorted multiset} \; \{p_{M_1}(m), p_{M_2}(m), \cdots, p_{M_k}(m)\}\,\}$$

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$$M'_1 \to M'_2 \to \cdots \to M'_q$$

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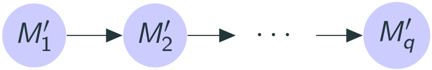

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Let $S = \{M_1, M_2, \cdots, M_k\}$ *be a set of* disjoint *stable matchings. Let* $\alpha_{j,S}$ *denote the stable matching obtained by matching each man* m_i *to* $p_{i,S}(m_i)$*, the* j*th woman in the sorted set* $P_S(m_i) = \{w_i | (m_i, w_i) \in M, M \in \alpha_{j,S}\}$*. Then, the stable matchings from the set* $C = \{\alpha_{1,S}, \alpha_{2,S}, \cdots, \alpha_{k,S}\}$ *forms a k-length chain* $\alpha_{1,S} \prec \alpha_{2,S} \prec \cdots \prec \alpha_{k,S}$ *of disjoint stable matchings.*

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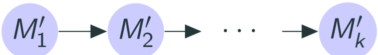

Maximum Size Set of Disjoint Stable Matchings

Theorem 8

For a given stable marriage instance, algorithm 3 gives the maximum size set of disjoint stable matchings.

Rotations

Rotations

For every stable matching M, we define the following:

$s_M(m)$

For any man m, let $s_M(m)$ denote the first women w on m's list such that w strictly prefers m to $p_M(w)$

$next_M(m)$

For any man m, let $next_M(m)$ denote $p_M(s_M(m))$

Note: $s_M(m)$ might not exist. Example: W_z.
Both $s_M(m)$ and $next_M(m)$ can be easily be found using *Reduced Lists*.

Roations

Definition of Rotations

An ordered list of matched pairs $\rho = (m_0, w_0), (m_1, w_1), \cdots, (m_{r-1}, w_{r-1})$ in a stable matching M is called as a rotation *exposed* in M if for each i $(0 \le i \le r-1)$, m_{i+1} is $next_M(m_i)$ where $i+1$ is taken modulo r.

Elimination of a Rotation

If M is a stable matching and $\rho = (m_0, w_0), (m_1, w_1), \cdots, (m_{r-1}, w_{r-1})$ is a rotation exposed in M, then M/æ is defined to be matching in which each man who is not in ρ stays married to his partner in M, and each man m_i in M is matched to $w_{i+1} = s_M(m_i)$

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Representation

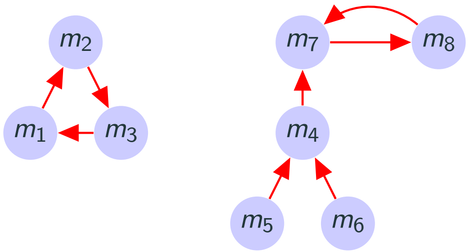

Figure 1: Graph $H(M)$

$(m_i, m_j) \in E(H(m))$ if $m_j = next_M(mi)$.

Properties of Rotations

- M/ρ is a stable matching such that $M \preceq M/\rho$
- Every stable matching except the women optimal matching has at least one rotation exposed in it.
- Every path from M_0 to M_z in $\mathcal{M}$ corresponds to some permutation of set of all rotations.

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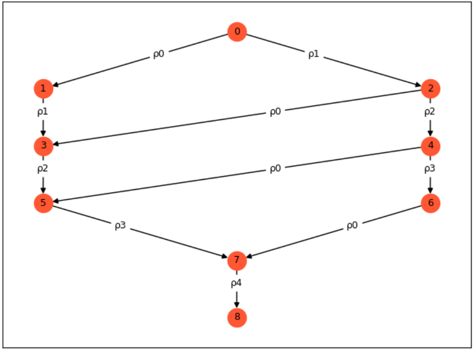

poset of rotations

The set of all rotations forms a partial order under the following relation.

$\rho_1 \prec \rho_2$ iff in *every* path from M_0 to M_z in $\mathcal{M}$ ρ_1 gets eliminated before ρ_2.

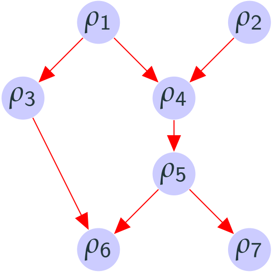

Figure 2: The Rotational Poset $\Pi((M))$

Theorem 9

There is a one-one correspondence between the closed subsets of $\Pi((M))$ *and stable matchings in* (M)

Theorem 10

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Algorithm based on $\Pi((M))$

Given a stable matching M_i, we can find the next best disjoint stable matching M_{i+1} by doing the following steps.

1. For every $(m, w) \in M_i$, find $R = \{\rho | (m, w) \in \rho\}$
2. Find $\hat{R}$ = closure of R
3. Return $M_{i+1} = M_i / \hat{R}$.

We use the above steps recursively (until we find a matching that intersects with M_z) to get the maximum chain of disjoint stable matchings.

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Future Work

- Disjoint Stable Matchings in the Roommate problem.
- When disjoint stable matchings do not exist, minimize pairwise intersection.

Thank You!