Research Statement

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My research has been focused on classical and quantum integrable systems and field theories and related areas of theoretical physics. Integrable systems have been of much interest across physics from the Hydrogen atom and the 2d Ising model to spin chains, sigma models and $\mathcal{N} = 4$ super Yang-Mills theory. Only a few physically realized systems are integrable. Neverthless, integrable models serve as toy models for investigating physical mechanisms such as the mass gap or holography. Moreover, we can study non-integrable systems as perturbations of integrable systems.

My thesis concerns the dynamics and integrability of a scalar field theory dual to the 1+1d SU(2) Principal chiral model (PCM) and its reduction to a mechanical system describing a class of nonlinear waves. The PCM serves as a toy-model for QCD as it is asymptotically free and displays a mass gap. Interestingly, the PCM is 'pseudo-dual' to a scalar field theory introduced by Zakharov and Mikhailov and Nappi that is strongly coupled in the ultraviolet and could serve as a toy-model for non-perturbative properties of theories with a Landau pole (such as $\lambda\phi^4$ in the Higgs sector of the standard model). In particular, one wishes to identify degrees of freedom appropriate to the description of the dynamics of such models at high energies. The dual scalar field theory is obtained by a non-canonical transformation of the principal chiral field. Thus, unlike the semi-direct product of $\mathfrak{su}(2)$ and abelian current algebras of the PCM, its pseudo-dual is based on a nilpotent current algebra. Theories that admit a formulation in terms of quadratic Hamiltonians and nilpotent Lie algebras are particularly interesting: they include the harmonic and anharmonic oscillators as well as field theories such as Maxwell, $\lambda\phi^4$ and Yang-Mills. Recently, Rajeev and Ranken (2016) obtained a mechanical reduction by restricting this nilpotent scalar field theory to a class of nonlinear constant energy density classical waves expressible in terms of elliptic functions. In other words, this model is a Hamiltonian system with 3 degrees of freedom describing nonlinear waves in a 1+1d scalar field theory dual to the SU(2) PCM. These novel 'screw-type' continuous waves could play a role similar to solitary waves in other field theories.

In [1], we have investigated the integrable features of this Rajeev-Ranken (RR) model. We study the Hamiltonian and Lagrangian formulations of the RR model and its classical integrability, identifying Darboux coordinates, Lax pairs, classical r-matrices and a degenerate Poisson pencil. We identify Casimirs as well as a complete set of conserved quantities in involution and the canonical transformations they generate. They are related to Noether charges of the parent nilpotent scalar field theory and are shown to be generically functionally independent, implying Liouville integrability. The singular submanifolds where this independence fails are identified and shown to be related to the static and circular (trigonometric) submanifolds of the phase space. We also find an interesting relation between this model and the Neumann model allowing us to discover a new Hamiltonian formulation of the latter.

In [2], we give the equations of the RR model a new interpretation as Euler equations for a centrally extended Euclidean algebra with a quadratic Hamiltonian. Thus, they bear a kinship to Kirchhoff's equations for a rigid body moving in a perfect fluid. Solutions of the RR model are also interpreted as a special family of flat $\mathfrak{su}(2)$ connections in 1+1 dimensions. Though analytic solutions in terms of elliptic functions had been found, deeper questions about the model's structure and integrability were open. In [2], we use the Casimirs of the Poisson algebra to find all symplectic leaves and Darboux coordinates on the six dimensional phase space of the model. The system is Liouville integrable on each symplectic leaf and the generic common level sets of conserved quantities are shown to be 2-tori. Going beyond the generic cases, we find three more types of common level sets: horn-tori, circles and points. They are related to places where the conserved quantities develop relations and to the degeneration of solutions from elliptic to hyperbolic and circular. An elegant geometric construction allows us to realize each common level set as a fibre bundle with base determined by the roots of a cubic polynomial. By contrast with the dynamics on tori and circles, which is Hamiltonian, that on horn tori is shown to be a gradient flow. In fact, horn tori behave like separatrices and are also associated to a transition in the topology of energy level sets. Finally, by a careful use of the Poisson structure and elliptic function solutions, we discover a family of action-angle variables for the model away from horn-tori. In addition, we have investigated the stability of classical static solutions of the RR model and the screw-type solutions of its parent field theory.

In [3], we show that the equations of motion (EOM) of the RR model can be interpreted in terms of a 3d cylindrically symmetric anharmonic oscillator. We exploit this to canonically quantize the model and separate variables in the Schrödinger equation. Though the radial equation is in general not exactly solvable, its analytic properties are studied and it is shown to be reducible to a generalized Lamé equation. We obtain the energy spectrum at weak coupling and its dependence on the wavenumber in a suitably defined strong coupling limit. We separate variables in the Hamilton-Jacobi equation and use this to find the WKB quantization condition. The EOM of the RR model can also be interpreted as Euler equations for a nilpotent Lie algebra. We exploit our canonical quantization to uncover an infinite dimensional reducible unitary representation of this nilpotent algebra which is then decomposed using its Casimir operators.

There are several directions of research that arise from this work. The possible extension of our results from the mechanical reduction to the scalar field theory is an interesting but challenging task. Other directions include a bi-Hamiltonian formulation of the RR model and quantizing the model using alternative approaches such as via path integrals or quantum R-matrices. Understanding the dynamics using the Jacobian of the spectral curve associated with our Lax formulation is another possible direction.

Refs. [4, 5] contain an exposition on Lax pairs and the zero curvature representation. The idea of realizing a nonlinear evolution equation as a compatibility condition between a pair of linear equations is explained by considering the examples of the harmonic oscillator, Toda chain, Eulerian rigid body, Rajeev-Ranken model, KdV and the nonlinear Schrödinger equations.

I have attended the Theoretical High Energy Physics SERB Main school (University of Delhi), Young Researchers Integrability School and Workshop: A modern primer for 2D CFT (ESI, Vienna), Conference on Nonlinear Systems and Dynamics (JNU, Delhi) and a school on Integrable systems in Mathematics, Condensed Matter and Statistical Physics (ICTS, Bangalore). Outside of integrable systems, I have also been exposed to a variety of topics of current interest in theoretical physics such as quantum entanglement, the SYK Model, nonlinear dynamics, fluid mechanics, PDEs, the WZW model and two-dimensional CFTs. I am open to exploring problems from these and related areas of physics.

References

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