TOPOLOGY: THE JOURNEY INTO SEPARATION AXIOMS

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Abstract. In this journey, we are going to explore the so called "separation axioms" in greater detail. We shall try to understand how these axioms are affected on going to subspaces, taking products, and looking at small open neighbourhoods.

1. What this journey entails

1.1. **Prerequisites.** Familiarity with definitions of these basic terms is expected:

- **Topological space**
- **Open sets**, **closed sets**, and **limit points**
- **Basis** and **subbasis**
- **Continuous function** and **homeomorphism**
- **Product topology**
- **Subspace topology**

The *target audience* for this article are students doing a first course in topology.

1.2. **The explicit promise.** At the end of this journey, the learner should be able to:

- Define the following: T_0, T_1, T_2 (Hausdorff), T_3 (regular), T_4 (normal)
- Understand how these properties are affected on taking subspaces, products and other similar constructions

2. What are separation axioms?

2.1. **The idea behind separation.** The defining attributes of a topological space (in terms of a family of open subsets) does little to guarantee that the points in the topological space are somehow distinct or far apart. The topological spaces that we would like to study, on the other hand, usually have these two features:

- Any two points can be separated, that is, they are, to an extent, far apart.
- Every point can be approached very closely from other points, and if we take a sequence of points, they will usually get to cluster around a point.

On the face of it, the above two statements do not seem to reconcile each other. In fact, topological spaces become interesting precisely because they are nice in *both* the above ways. The first of these is related to the concern of "separation axioms", and this is what we will look at here.

2.2. **The concept of quasiorder.**

Definition. A **quasiorder**(defined) or **preorder**(defined) is a reflexive and transitive relation on a set. Under a quasiorder $\leq$, two elements a and b are said to be **equivalent**(defined) if $a \leq b$ and $b \leq a$. The above notion of *equivalence* partitions the set into equivalence classes, and the quasiorder boils down to a partial order (that is, a reflexive antisymmetric transitive relation) on the equivalence classes.

Here are some examples of quasiorders.

Suppose we have a collection of people. We say that a person a is *not older than* a person b if the age of a is less than or equal to the age of b (the age is simply counted in years, with no fractions). Clearly, this relation is reflexive: each person is not older than himself or herself. Moreover, it is also transitive: if a is not older than b, and b is not older than c, then the age of a is less than or equal to the age of c.

Under this quasiorder, if $a \leq b$ and $b \leq a$, we cannot conclude that a and b are the same person. What we can conclude is that they have the same age. In fact, "having the same age" is precisely the equivalence relation for the quasiorder, and each equivalence class is represented by a value of age.

Among these equivalence classes, there is a total ordering. That is, if two people have different ages, then either one is not older than the other, or the other is not older than the first.

Here's another example. Suppose we associate to every family a collection of wealth indices. These give vectors in $\mathbb{R}^n$, which we'll call the wealth vector of the family. We say that a family a is *not better off in any respect* than a family b if every coordinate of the wealth vector of a is less than or equal to the corresponding coordinate in b. Clearly, the relation is reflexive and transitive. However, it may not be antisymmetric – if two families a and b have the same wealth vector, we have $a \leq b$ and $b \leq a$.

In this case, then, every wealth vector gives rise to an equivalence class of families. Among these equivalence classes, the relation is a partial order, in fact, it is the partial order of coordinate wise dominance.

Concept Testers

(1) Prove that a quasiorder on a set defines a partial order on its equivalence classes (where an equivalence class is a set of elements such that for any a and b in it, $a \leq b$ and $b \leq a$).
(2) A discrete partial order is a partial order where no two distinct elements are comparable. Prove that if a quasiorder is such that $a \leq b \iff b \leq a$, then the associated partial order is the discrete partial order.
(3) Prove that if, for a given quasiorder, every a and b satisfy either $a \leq b$ or $b \leq a$, then the associated partial order is the total order.

2.3. Quasiorder by closure.

We'll begin by proving the following result:

Claim. Given a topological space S, consider the relation $a \leq b \iff a \in \overline{b}$. (That is, every open set about a contains b). Then this relation is a quasiorder, that is, it is reflexive and transitive.

Proof. Clearly, $a \in \overline{a}$, so the relation is reflexive. For transitivity, observe that suppose $a \leq b$ and $b \leq c$. Then every open set about a contains b, and and every open set about b contains c. But this clearly means that every open set about a contains c, and hence $a \in \overline{c}$, thus $a \leq c$. ☐

In some sense, a *sticks* to b – whatever open set about a we take, that open set must also contain b. So, a is in some ways dependent or inferior to b.

Now, if a and b are *equivalent* under this quasiorder, that is, we have both $a \leq b$ and $b \leq a$, then that means that every open set contains either *both* a and b or *neither* a nor b. This means that any local behaviour around a is also around b and vice versa. In this sense, we call the two points indistinguishable.

Definition. If $a \in \overline{b}$ and $b \in \overline{a}$ the points a and b are said to be **indistinguishable**(defined).

It is clear that indistinguishability is an equivalence relation (see above) We are now motivated to a definition:

Definition. A topological space is said to be T_0(defined) or **Kolmogorov**(defined) if no two distinct points are indistinguishable. In other words, a topological space S is called T_0 if given any two points a and b, there is either an open set containing a but not b, or an open set containing b but not a.

Concept Testers

(1) Consider a space with two or more points, with the **discrete topology**(recalled), that is, with the topology under which all subsets are open. What is the quasiorder in this topology? Is it T_0?
(2) Consider a space with two or more points with the **trivial topology**(recalled), that is, the only open subsets are the whole space and the empty space. What is the quasiorder in this case? (That is, when is it true that $a \leq b$?) Is it T_0?
(3) Prove that the quasiorder by closure for a T_0 topological space is a partial order.

Points to Ponder

- Given an arbitrary topological space, consider a new topological space whose points are equivalence classes under the quasiorder by closure. What topology should be given to this topological space, so that the quotient map (taking each element of the original topological space to its equivalence class) is continuous, and the new topological space is T_0? A natural way of giving such a topology is the so called Kolmogorov quotient construction.
- Given a partially ordered set, can we find a T_0 topological space such that the quasiorder by closure is the given partially ordered set? Is this topological space unique for the purpose?

2.4. **The T_1 assumption.** A T_0 space is a space where no two points can be indistinguishable. We shall see later that given any topological space, we can pass to an associated T_0 space by a quotient construction (called the **Kolmogorov quotient**(first used)). Thus, we can always get started by working with a T_0 topological space.

Nice though they are, T_0 topological spaces can still have some points sticking to others. So, it is quite possible that $a \in \bar{b}$, in other words, it is possible that every open set containing a contains b as well. All we know is that we cannot have a sticking to b as well as b sticking to a.

An even better *separability* assumption is that no point sticks to any other point. In other words, $a \leq b$ never happens unless $a = b$. Another way of saying this is that the closure quasiorder is trivial.

Definition. A topological space is said to be T_1(defined) if the following equivalent conditions hold:

- Given any two points $a \neq b$ in it, there is an open set containing a but not b, and there is an open set containing b but not a.
- No point lies in the closure of any other point.
- In the quasiorder by closure, no two elements are comparable. In other words, the quasiorder is the *discrete* partial order.
- Every point is its own closure. In other words, all one point subsets are closed.

The last way of characterizing T_1 topological spaces is that every point is a closed set. Let's try to understand this visually. When one point sticks to the other, then that other point is actually open to this point – it has a boundary that interfaces with this point. We now want to avoid this altogether.

Concept Testers

(1) A topological space is said to be **symmetric**(defined) if whenever a and b are two points, a lies in the closure of b if and only if b lies in the closure of a. Prove that a symmetric topological space can also be defined as follows: given any two points a and b in the space, there is an open set containing a and not b, if and only if there is an open set containing b but not a.
(2) Prove that T_1 spaces are precisely the symmetric T_0 spaces.
(3) Prove that a topological space is symmetric if and only if the quasiorder by closure gives rise to a discrete partial order on the equivalence classes.
(4) A topological space is said to be **homogeneous**(defined) if given any two points in the topological space, there is a homeomorphism of the topological space taking one to the other. In other words, any two points look the same to the topological space. Prove that every homogeneous topological space is symmetric.
(5) Prove that given a topological space X and a subspace Y, the quasiorder by closure on Y is simply the same as the quasiorder by closure on X. In other words, if $x, y \in Y$ and $x \leq y$ in X, then $x \leq y$ in Y and vice versa.

    Hence show that every subspace of a T_0 space is T_0, every subspace of a symmetric space is symmetric, and every subspace of a T_1 space is T_1.
(6) Prove that every finite T_1 space is discrete. Hence, show that every finite subspace of a T_1 space is discrete.
(7) Prove that if we add more open sets to the topology, that is, we pass from a coarser to a finer topology, T_0 spaces remain T_0, and T_1 spaces remain T_1. In fact, for most separation axioms , addition of more open sets only increases the extent of separation, rather than decreasing it.

Pause and Recollect

(1) Rewrite the definitions of T_0, symmetric and T_1 each in these three ways: using open set separation, using closures, and using the properties of the quasiorder by closure. Try to write the definitions such that for each way of writing, it is clear that T_1 spaces are precisely the symmetric T_0 spaces.

Points to Ponder

- Suppose X is a topological space, and a group G acts on X in such a way that for every $g \in G$, the induced map $X \to X$ is continuous. Prove that X is a homogeneous space. Hence, show that X is symmetric.
- Given a group G, consider the space X whose points are the normal subgroups of G. Define the following subsets as closed: the set of all normal subgroups of G containing a given normal subgroup N. Is an arbitrary intersection of these "closed" sets closed? Is a finite union of these "closed" sets closed?

- What happens if instead of taking X as the space of normal subgroups, we take X as the space of normal subgroups N such that whenever $H_1 \cap H_2 \leq N$ then either $H_1 \leq N$ or $H_2 \leq N$? Do we then get a topology? Is it T_0? Is it symmetric?
- Is it true that every subspace of a homogeneous space is homogeneous?
- Is it true that if a topological space is symmetric, then passing to a finer topology will retain the property of being symmetric?

3. Separation of points and subsets

3.1. The separation setup. We'll see a whole lot of definitions now, some of which may seem ill motivated to begin with. So I'll try to describe the general setup. This general setup may not seem to make much sense right now, but after things unfold a little bit, we shall get back to it with a better understanding. Suppose we have a separation property. There are two players – a *spoiler* who is trying to prove that the topological space does *not* satisfy the property, and a *prover* who wants to show that it does.

- The spoiler starts with a pair of subsets that are separate, or far apart.
- The prover now tries to find bigger subsets that expand the original pair, but are still far apart in some sense.

If the prover, can always win *regardless* of what the spoiler chooses, then the topological space is said to have the property.

Here are some criteria to say that two subsets are separate:

- They are **disjoint**(defined) – that is, their intersection is empty.
- They are **separated**(defined) – that is, the intersection of each of them with the closure of the other is empty.
- They are **closure disjoint**(defined) – that is, their closures are disjoint.
- They are **function separated**(defined) – that is, there is a continuous function taking 0 on one subset and 1 on the other.

Concept Testers

(1) Recall that for a continuous function between topological spaces, the inverse image of a closed set is closed. Using this fact, prove that the inverse images of 0 and 1 under any continuous function from a topological space to $[0, 1]$ are closed. Hence, prove that if two subsets are function separated, then they are closure disjoint.
(2) Prove that function separated $\implies$ closure disjoint $\implies$ separated $\implies$ disjoint.

Points to Ponder

- Is moving to a finer topology advantageous for the spoiler or for the prover? That is, if there are more open sets, who stands to gain?
- Is moving to a subspace advantageous for the spoiler or for the prover? That is, who stands to gain by passing from the topological space to a subspace?

3.2. The T_0 and T_1 conditions. What sense can we make of the T_0 and T_1 conditions in the spoiler-prover setup? For the T_0 condition:

- The spoiler simply picks two points.
- The prover then picks one of the points of *his own* choice, and an open set about that point that does not contain the other point.

The T_1 condition, on the other hand, translates to the following:

- The spoiler picks two points and marks one of them.
- The prover then gives an open set containing the marked point but not the other point.

Another way of putting the T_1 condition is:

- The spoiler picks two points.
- The prover finds, for each of the points, an open set containing that point and not the other one.

It is clear from this definition that T_1 is stronger than T_0, because in the T_1 case, the work of proving is a little harder.

Pause and Recollect

(1) How is the subspace topology defined? How did we show that every subspace of a T_1 space is T_1 in the subspace topology?

4. On Hausdorff spaces

4.1. **Hausdorff spaces.** The condition which we call "Hausdorffness" now was assumed by Felix Hausdorff as one of the fundamental axioms of topological spaces. For many topologists, topological spaces that are not Hausdorff are not worth studying at all.

Definition. A topological space is said to be **Hausdorff**(defined) if given any two distinct points x and y in the space, there are open set U_1 and U_2 such that U_1 contains x, U_2 contains y, and $U_1 \cap U_2 = \emptyset$.. In other words, any two distinct points can be separated by disjoint open sets.

Hausdorffness corresponds to the situation where the spoiler can pick on any pair of *distinct points*, and the prover must expand them to *disjoint open subsets*. The intuitive importance is as follows: an open set about a point represents the notion of being *close* to the point. When two points are separated by disjoint open sets, that means that there is nothing that can come quite close to *both* of them.

We now prove a basic result:

Claim. Every subspace of a Hausdorff space is Hausdorff.

Proof. Suppose X is a Hausdorff space, and A is a subspace of X. The spoiler now picks two distinct points x and y in A, and the prover must demonstrate that there are disjoint sets containing x and y respectively that are open in the subspace topology for A.

Since x and y are distinct points of X, the prover has at his disposal disjoint open sets of X, namely U and V, such that U contains x and V contains y. He them picks $U \cap A$ and $V \cap A$ as subsets of A. Clearly:

- $x \in A$ and $x \in U$, so $x \in U \cap A$. Similarly, $y \in V \cap A$.
- U and V themselves are disjoint, so $U \cap A$ and $V \cap A$ are disjoint.
- $U \cap A$ is open relative to A, because it is the intersection with A of an open set in X. Similarly $V \cap A$ is also open relative to A.

Hence, the prover has managed to pick disjoint open sets containing x and y, relative to the subspace topology of A.

□

Concept Testers

(1) Prove that every Hausdorff space is T_1. Use the open set definition of T_1.
(2) Prove that passing to a finer topology retains the property of being Hausdorff. (Recall that a finer topology is one with more open sets, but with the earlier open sets retained).
(3) A topological space is said to be **conditionally Hausdorff**(defined) if given any two points, such that there is an open set containing one but not the other, there are disjoint open sets separating them. Prove that a conditionally Hausdorff T_1 space is precisely the same as a Hausdorff (or T_2) space.

4.2. **Sequences and limits.** We are already familiar with the notion of limits of sequences from calculus. The notion is now defined for an arbitrary topological space:

Definition. A **sequence of points**(defined) in a space X is a function $\mathbb{N} \to X$. The function may not be injective. The image of $n \in \mathbb{N}$ is typically denoted as x_n, or a_n.

Given a sequence of points $\{x_n\}_{n\in\mathbb{N}}$ in a space X, a point $x \in X$ is said to be a **limit**(defined) of x_n (or, $x_n \to x$) if given any open set U containing x, *all but finitely many* n satisfy $x_n \in U$.

This is equivalent to the usual statement that there exists an n_0 such that $x_n \in U$ for $n \geq n_0$. Intuitively, we can think of it as follows: however small an open cage we construct about x, the sequence of points x_n eventually enters (or gets trapped in) that open cage.

The notion of "limit" of a sequence is somewhat stronger than the notion of "limit point" of a set. Note that for the limit of a sequence, we insist that for any neighbourhood, *all but finitely many* points lie inside. Whereas, in the case of limit points, all we insist is that for any neighbourhood there is some element of the set in that neighbourhood.

Claim. The limit of a sequence is a limit point of the associated set, if it is not itself a member of that set. In general, any limit of a sequence is a limit point unless the sequence eventually becomes constant at that point.

Proof. Let x_n be a sequence in X and x be a limit of the sequence. We want to show that every open set U containing x must contain some $x_n \neq x$.

Clearly, every open set about x contains all the x_n for all but finitely many n. Because the sequence does not become eventually constant, it cannot happen that all $x_n = x$. Thus, there is at least one n for which $x_n \neq x$. This completes the proof. □

Basically, the notion of limit point means that the sequence comes arbitrarily close to the point infinitely often. Whereas the notion of limit means that the sequence keeps coming closer and closer to the point, and gets *eventually trapped* in any cage about the point, however small. Thus, a sequence that keeps oscillating between 0 and 1, or even a superposition of sequences that converge to 0 and 1, has both 0 and 1 as limit points but neither as a limit.

Concept Testers

(1) Prove that if $f : X \to Y$ is a continuous map and x_n is a sequence in X converging to x, then $f(x_n)$ converges to $f(x)$.

Points to Ponder

- Find an example of $f : X \to Y$ and a sequence in X that has no limit, but whose image under f has a limit.

4.3. **Sequences in Hausdorff spaces.** We prove a very important result for Hausdorffness:

Claim. In a Hausdorff space, every sequence has at most one limit.

Proof. Let X be a Hausdorff space and x_n be a sequence. Let y and z be two limits of x_n. By Hausdorffness of X, there are disjoint open sets U and V containing y and z respectively. Now, there are only finitely many n for which $x_n \notin U$, and only finitely many n for which $x_n \notin V$. Thus, there are only finitely many n for which $x_n \notin U \cap V$. But this means there is some $x_n \in U \cap V$, contradicting the fact that U and V are disjoint. □

The essential idea behind the proof is this: make small open cages about x and y, Eventually, all points in the sequence will enter the open cage about x, but they'll also eventually enter the open cage about y. However, these open cages are disjoint! So basically, the sequence cannot enter both of them forever and it must make up its mind.

A topological space where every sequence has at most one limit is called a **US topological space**(defined). This is not a very standard term, and we are just using it for a bit of convenience.

Concept Testers

(1) Prove that every US topological space is T_1.

Points to Ponder

- We know that Hausdorff $\implies$ US $\implies$ T_1. Prove that each of these implications is strict.

4.4. **Morphic characterization of T_1 and Hausdorff.** Given a set map $f : X \to Y$, the fibers of the map are the inverse images of singleton sets. We now show that:

Claim. A topological space Y is T_1, if and only if for every continuous map $f : X \to Y$, the fibers are closed.

Proof. Suppose $f : X \to Y$ is continuous, and Y is T_1. We claim that the fibers are closed: the fibers are the inverse images of one point sets in Y. Y being T_1, each of these one point sets is closed. And the inverse image of a closed set under a continuous function is closed. Hence, the fibers are closed.

To prove the converse, observe that if the fibers are closed for *every* continuous function, we can simply take the function to be the identity map $f : Y \to Y$. Here, the fibers are just the points of Y, and if these are closed, then clearly Y is T_1. □

Hence, the corollary:

Claim. If X is T_1, and a continuous function takes a subset A of X to a point x, it also takes the closure of A to the point x.

However, with Hausdorffness, we can do a little better, namely that:

Claim. If A is a dense subset of X and Y is a Hausdorff space, then every continuous function from A to Y extends in at most one way to a continuous function from X to Y.

Notice that this is somewhat better than the previous result, which had to assume the function was constant. This improvement is possible because of the status upgrade of Y from T_1 to T_2.

Proof. Suppose there are two different extensions of f to X. That is, there are two distinct functions $f_1 : X \to Y$ and $f_2 : X \to Y$ such that $f_1|_A = f_2|_A = f$. Then, suppose $x \in X$ with $f_1(x) \neq f_2(x)$. Since Y is Hausdorff there are open subsets about $y_1 = f_1(x)$ and $y_2 = f_2(x)$. Call these V_1 and V_2. Then look at $U_1 = f_1^{-1}(V_1)$ and $U_2 = f_2^{-1}(V_2)$. Now U_1 and U_2 are open sets, so $U_1 \cap U_2$ is an open set, and it contains x. Hence, it intersects A (because A is dense in X). Let $a \in A$ be a point in $U_1 \cap U_2$.

Now, $f_1(a) \in V_1$ and $f_2(a) \in V_2$. But as $a \in A$, $f_1(a) = f_2(a) = f(a)$. Thus, $f(a) \in V_1 \cap V_2$. However, by our original assumption V_1 and V_2 were *disjoint*. This gives the contradiction. □

The basic idea behind the proof is that if a point in the space has two different possible images, then making open cages about those images, we can get a point in the *dense* subset whose image lies simultaneously in *both the open cages*. This gives the argument.

Notice that what we are really using these open cages for is to *separate* the images nicely. And to construct these open cages, we require the image space to be Hausdorff.

Concept Testers

(1) Use the above result to prove the following variant: "if A and B are subsets of X with B contained in the closure of A, and Y is Hausdorff, then any continuous function from A to Y can be extended to a continuous function from B to Y in at most one way".
(2) A **fixed point**(defined) if a function from a set to itself is a point that equals its own image. That is, for $f : S \to S$, x is a fixed point if $f(x) = x$. Prove that the set of fixed points of a continuous map from a Hausdorff topological space to itself is a closed set.

Points to Ponder

- Let G be an Abelian group and S a set. Define the period group of a function $f : G \to S$ as the set of all $h \in G$ such that $f(x + h) = f(x)$ for every $x \in G$. f is **periodic**(defined) if the period group is nontrivial, and **fundamentally periodic**(defined) if the period group is infinite cyclic. The generator of the infinite cyclic group is a fundamental period.

  Prove that every continuous periodic function from $\mathbb{R}$ to a Hausdorff space is fundamentally periodic.

4.5. Sober spaces and irreducible subsets.

Definition. A topological space is said to be **irreducible**(defined) if it cannot be expressed as the union of two proper closed subsets.

Equivalently, a topological space is irreducible if it does not contain a pair of disjoint nonempty open sets. Clearly, irreducible spaces are far from Hausdorff. However, irreducible spaces are not as uncommon as they may seem. In fact, it is readily check that:

Claim. The closure of every one point set is irreducible.

Proof. Let $A \subseteq X$ be the closure of a point x. Suppose $A = A_1 \cup A_2$ with both A_1 and A_2 proper closed subsets of A. A closed subset of a closed subset is closed. Hence, both A_1 and A_2 are closed subsets of X strictly contained in A. But one of these must contain x, contradicting the fact that A is the smallest closed set containing x. □

The one point can be thought of as a kind of "witness for irreducibility".

And now here is a definition:

Definition. A topological space is said to be **sober**(defined) if the only irreducible closed subspaces are the closures of one point sets.

In other words, whenever we encounter an irreducible closed subset, there is a single point that provides a witness for the irreducibility.

Of particular interest are sober T_1 spaces, that is, spaces where the only irreducible closed subsets are the one point sets.

Concept Testers

(1) Prove that there are only two possibilities for an irreducible Hausdorff space: the one point space, and the empty space.

(2) Prove that a Hausdorff space is always a sober T_1 space. In fact, in a Hausdorff space, the only irreducible subspaces are one point sets. (in contrast to sober T_1 spaces where we can only guarantee that all irreducible closed sets are one point sets).

Points to Ponder

- Are there sober T_1 spaces that are not Hausdorff? What about sober T_0 spaces that are not T_1?
- Can we reformulate the definition of "sober" in the framework of the quasiorder by closure?

4.6. **Locally Hausdorff spaces.** The term "locally" is typically used to indicate that every point has an open neighbourhood with the property.

Definition. A topological space is said to be **locally Hausdorff**(defined) if every point has an open neighbourhood that is a Hausdorff space.

Clearly every Hausdorff space is locally Hausdorff, because for the "open neighbourhood", we can take the whole space itself. We can also take any other open neighbourhood, which will be Hausdorff by the hereditary nature of the property of being Hausdorff.

What about locally T_1 and locally T_0? Well, they are the same as T_1 and T_0 respectively.

Claim. If every point in a topological space has an open neighbourhood that is T_1, then the whole space is T_1.

Proof. The spoiler picks two points x and y, and asks the prover to convince him that there is an open set containing x but not y. The prover first picks an open neighbourhood U of x that is T_1. There are two cases:

- If $y \notin U$, then U itself does the prover's job.
- Otherwise, $y \in U$. Then, there is a set V containing x but not y, open in U. Because an open set of an open set is open, V is an open set in X containing x and not y, and hence V does the job.

□

Concept Testers

(1) Prove that every locally T_0 space is T_0.
(2) Prove that every locally Hausdorff space is T_1 (or equivalently, locally T_1).
(3) Prove that Hausdorffness is equivalent to the following property: "given any two points, there is an open set containing both of them that is Hausdorff in the subspace topology".

Points to Ponder

- A topological space property π is called hereditary if every subspace of a space having π, also has π. Given a hereditary property π, a topological space is termed locally π if every point has a neighbourhood with property π. (Actually, we don't need to assume π is hereditary, but it is convenient for our purposes here). A topological space property π is called hereditary and local if it is hereditary and π = locally π. Which of the separation properties studied so far are hereditary? Which are local? (We have seen T_0, T_1, T_2, US, sober, irreducible, symmetric, homogeneous).
- Prove that if $a \implies b$, then locally $a \implies$ locally b.
- If a is hereditary, is locally a also hereditary?

4.7. **Collectionwise Hausdorffness.**

Definition. A topological space is said to be **collectionwise Hausdorff**(defined) if it is T_1 and given a discrete subset of the topological space (that is, a subset on which the induced topology is discrete there are open sets about every point in the discrete subset, that are pairwise disjoint..

As we had proved long ago, every finite subset of a T_1 space is discrete. Thus, in particular, every two point subset is discrete. Thus, a collectionwise Hausdorff topological space must be Hausdorff.

Are Hausdorff spaces collectionwise Hausdorff? Not necessarily. However, we can say the following for Hausdorff spaces in general:

Claim. Given any finite collection of distinct points $x_1, x_2 \dots x_n$ there are open sets $U_1, U_2 \dots U_n$ such that U_i contains x_i and $U_i \cap U_j$ is empty for $i \neq j$.

Proof. For each pair $i \neq j$, let V_{ij} and V_{ji} be disjoint open sets containing x_i and x_j respectively. Then, consider U_i to be the intersection of all V_{ij} with $j \neq i$. Clearly, we have:

- U_i, being a finite intersection of open sets, is open.
- $x_i \in V_{ij}$ for all j, so $x_i \in U_i$.
- For any i and j, $U_i \subseteq V_{ij}$ and $U_j \subseteq V_{ji}$ with $V_{ij} \cap V_{ji}$ being empty. Thus U_i and U_j are disjoint.

So, this choice of U_i works!

□

The basic idea: separate them pairwise, and then take the intersection of all the open sets used to separate. The problem with extending this to arbitrary discrete sets is that we may need to take an infinite intersection of open sets, which does not make sense.

Points to Ponder

- A topological space is said to be **hereditarily collectionwise Hausdorff**(defined) if every subspace of it is collectionwise Hausdorff. We know that Hausdorff is the same as hereditarily Hausdorff. Is collectionwise Hausdorff the same as hereditarily collectionwise Hausdorff?
- Is the property of being collectionwise Hausdorff the same as the property of being locally collectionwise Hausdorff?

4.8. **The diagonal is closed.** Here is yet another characterization of the property of being Hausdorff!

Claim. A topological space X is Hausdorff if and only the diagonal is closed in $X \times X$ with the product topology.

Proof. Let D denote the diagonal $\{(x, x)|x \in X\}$ in $X \times X$.

Suppose D is closed. Then the complement of D is open. We want to show that X is Hausdorff.

Suppose the spoiler picks $x \neq y$. The point (x, y) lies in an open set disjoint from D. In particular, there is a basis open set about (x, y) that does not intersect D. Let $U \times V$ be such a basis open set. (so U and V are both open in X). Clearly, if $y \in U \cap V$, then $(y, y) \in U \times V$. But $U \times V$ does not intersect D, and hence U and V are disjoint open sets. Thus, the prover can respond by giving U and V as the disjoint open sets containing x and y.

Conversely, suppose X is Hausdorff. Then, we want to show that D is closed.

We know that given $x \neq y$, there are disjoint open sets U and V containing x and y respectively. Thus, any (x, y) lies inside an open set $U \times V$. Further, as $U \cap V$ is empty, the set $U \times V$ does not intersect D. Hence, for every point outside D, there is a neighbourhood of the point outside D. Taking the union of all these neighbourhoods, we conclude that the complement of D is open, and hence that D is closed. □

Though the two directions of proof seem almost identical to one another, there is a subtle difference. In proving that the diagonal is closed from Hausdorffness, all we need to use is that products of open sets are open. Thus, the same proof will go through if we add more open sets. On the other hand, the proof of Hausdorffness from the diagonal being closed critically uses the fact that the so called *open rectangles* (the products of open sets) form a *basis*.

4.9. **A summary.** We have seen *three* primary separation properties: the properties of being T_0, T_1 and T_2. Each of these can be described in terms of spoiler-prover games as follows:

- For T_0, the spoiler picks two distinct points. The prover then chooses one of them and exhibits an open set containing that but not the other.
- For T_1, the spoiler picks two distinct points. For each point, the prover finds an open set containing that point but not the other.

  Alternatively, the spoiler marks one of the points and the prover is required to find an open set containing that but not the other.
- For T_2, the spoiler picks two distinct points. The prover finds disjoint open sets containing the two points.

We have so far seen the following:

Primary property	Hereditary version	Local version
T_0 (Kolmogorov)	T_0	T_0
T_1	T_1	T_1
T_2 (Hausdorff)	T_2 (Hausdorff)	locally Hausdorff

Pause and Recollect

(1) Convince yourself of the results stated in the table above.

(2) List all the equivalent formulations of the property of Hausdorffness. How have these formulations been shown to be equivalent?
(3) Recall the definitions of the "product topology" and the "box topology" and convince yourself that these definitions have been used correctly in the "diagonal is closed" proof.

5. Products of the spaces

5.1. **Product of Hausdorff spaces is Hausdorff.** To recall a bit about products (a process we set in motion with the "diagonal is closed" proof) let us first prove that a product of two Hausdorff spaces is Hausdorff.

Claim. The product of Hausdorff spaces is Hausdorff in the product topology.

Proof. Let X and Y be the two Hausdorff spaces. Then, the product space is $X \times Y$.

The *spoiler* can now begin by choosing points (x_1, y_1) and (x_2, y_2). Notice that the spoiler needs to give the guarantee that the points are distinct: that is, he must ensure that either $x_1 \neq x_2$ or $y_1 \neq y_2$.

The prover now adapts his strategy accordingly. If $x_1 \neq x_2$, he first separates x_1 and x_2 in X. That is, he picks disjoint open sets in X: U_1 containing x_1 and U_2 containing x_2. Clearly $U_1 \times Y$ and $U_2 \times Y$ are disjoint open sets containing (x_1, y_1) and (x_2, y_2) respectively. Thus, the prover has succeeded.

If $y_1 \neq y_2$, he separates y_1 and y_2 in Y, say, by V_1 and v_2. Then $X \times V_1$ and $X \times V_2$ are disjoint open sets containing (x_1, y_1) and (x_2, y_2), and that is what the prover picks. □

A pictorial view of the above proof might make it clear. Suppose X is drawn on the horizontal and Y on the vertical axis. If the spoiler picks two points in $X \times Y$, it may happen that they have the same X coordinate (That is, they are in the same vertical line) or that they have the same Y coordinate (that is, they are in the same horizontal line). However, both the things cannot happen simultaneously. If they are not in the same horizontal line. we can separate the Y parts and take the corresponding cylinders. If they are not in the same vertical line, we can separate the X parts and take the corresponding cylinders.

From the game point of view, the prover has essentially exploited the *weak* coordinate of the spoiler – the one where the two points differ, and used that coordinate to obtain the separation.

Concept Testers

(1) Give a precise proof of the equivalence of the two definitions of regularity (assume the T_1 condition, though it is hardly relevant for us).
(2) Prove that a product of two T_1 spaces is T_1.
(3) Prove that a product of two T_0 spaces is T_0.
(4) Prove that if the product of two topological spaces is Hausdorff, then they both must be Hausdorff. (In fact, just view both of them as subspaces of the product). State and prove the analogous results for T_0 and T_1.
(5) Prove that the product of two locally Hausdorff spaces is locally Hausdorff.

Points to Ponder

- A property of topological spaces is said to be **closed under finite products**(defined) if the product of finitely many topological spaces with the property also has the property. Consider the general separation setup using the spoiler and prover. Which properties would naturally appear to be closed under finite products?
- Is it generally true that if a hereditary property π is closed under finite products, so is the property locally π?
- Is a finite product of collectionwise Hausdorff spaces collectionwise Hausdorff?

5.2. **Arbitrary products: box versus product topologies.** For arbitrary products, there are two ways of defining the topology:

- In the **box topology**(recalled), the basis for open sets are the **boxes**(recalled) – those sets that are obtained as the product of open subsets of each of the spaces.
- In the **product topology**(recalled), the basis for open sets are the **generalized cylinders**(recalled) – those sets that are products of open subsets of each space, with only finitely many of the open sets being proper. Generalized cylinders can also be thought of as finite intersections of cylinders (which can be thought of as forming a subbasis).

Every box is a generalized cylinder, but the converse is not true. Hence, the box topology is strictly finer than the product topology. In particular, if a product space is T_0, T_1 or Hausdorff in the product topology, it will be so in the box topology as well.

So we'll modify the proof given for finite products, to establish the following:

Claim. An arbitrary product of Hausdorff spaces is Hausdorff in the product topology (and hence, also in the box topology).

Proof. We will in fact show that given any two points in an arbitrary product of Hausdorff space, there are disjoint open cylinders separating them.

Once the spoiler has picked the points, the prover does the following:

- Locate a coordinate that is different for the two points.
- Finds disjoint open sets separating the values for that coordinate.
- Takes the open cylinders over these disjoint open sets, as the disjoint open sets in the product space.

□

Concept Testers

(1) Prove that an arbitrary product of T_1 spaces is T_1.
(2) Prove that an arbitrary product of T_0 spaces is T_0.

Points to Ponder

- Is an arbitrary product of locally Hausdorff spaces locally Hausdorff?

6. Regularity

6.1. Regular spaces.

There is a little ambiguity in topological circles as to the definition of regularity. This ambiguity arises from the question of whether to impose the T_1 condition(that is, the condition that points are closed. The notions of regularity are somewhat different with and without these assumptions:

Definition. A topological space is said to be **regular**(defined) if given a closed set C in the topological space and a point $x \notin C$,there are open sets U_1 and U_2 such that $C \subseteq U_1$, $x \in U_2$, and $U_1 \cap U_2 = \emptyset$.

A topological space is said to be $T_{3\text{(defined)}}$ if it is regular and points are closed.

In most texts, the terms regular and T_3 are considered equivalent, that is, when we use the term "regular", we assume that points are closed. Unless otherwise stated, we shall also assume this. That is, we shall assume that the term "regular" stands for 'regular and T_1".

Here's an alternative definition of regularity:

Alternative Definition. A topological space is said to be **regular**(defined) if given a point and an open neighbourhood of it, there is a neighbourhood of the point whose closure lies completely within the given open neighbourhood. That is, a topological space X is termed regular if given $x \in U \subseteq X$ with U open, there is an open set $V \ni x$ such that $\overline{V} \subseteq U$.

The two definitions are equivalent because of the following. Suppose a topological space is regular in the first sense. Then, given a point x and an open set U containing it, take C to be the complement of U. Then, by the definition, there are disjoint open sets U_1 and U_2 containing x and C respectively. Set V to be U_1. Clearly, there is an open set containing C disjoint from V, so $\overline{V}$ cannot contain any point in C. Thus $\overline{V} \subseteq U$.

We can do similar reasoning to derive the first definition from the second.

Concept Testers

(1) Prove that every regular T_1 space is Hausdorff.
(2) Can we replace the T_1 assumption by the T_0 assumption and still end up with the same meaning? That is, is every regular T_0 space a regular T_1 space?

6.2. Metaproperties of regularity.

Recall that for $\pi = T_0$ and $\pi = T_1$, we had:

- Every subspace of a π space is a π space.
- Every space that is locally π, is π.
- An arbitrary product of π spaces is a π space.

For Hausdorffness (that is, the T_2 condition) the first and third continued to hold, but it was no longer true that every locally Hausdorff space is Hausdorff.

Which of the results holds for regularity? The way out, of course, is to *carefully inspect* the proof given for Hausdorffness, and see what changes need to be made there. Note the fundamental difference between Hausdorffness and regularity: in Hausdorffness, any two distinct *point* are separated by disjoint open sets, and in regularity, any point and closed set are separated by disjoint open sets.

Recalling the proof that every subspace of a Hausdorff space was Hausdorff, we went along these lines:

- The spoiler picked two points in the subspace.
- The prover then looked at them as two distinct points in the whole space.
- They were separated by disjoint open sets in the whole space.
- These were then intersected with the subspace to give disjoint open subsets for the subspace, in the relative topology.

Now, here we have a point and a closed set, such that the point and the closed set do not intersect. So if we try to reason along the same lines as above, the problem we run into is at the transition from the first to the second step. Suppose $Y \subseteq X$ and $x \in Y$, C is closed in Y. Then x is a point in X, and C is a subset of X, but C may *not* be closed. We know, however, that there is a D closed in X whose intersection with Y is C.

What we also need to argue is that D cannot contain x. This is obvious – because the only points of Y in D are in C, and C does not contain x. So the proof goes through, and we have the following result.

Claim. Every subspace of a regular space is regular.

Proof. Let X be the regular topological space, and A a subset. The spoiler picks $x \in A$ and C closed in A. The prover now looks at x as a point in X, and picks D a closed subset of X such that $D \cap A = C$. Such a D exists by the way the subspace topology is defined. Clearly, whatever D is picked up for the purpose, x cannot lie in D because the only points in $D \cap A$ are in a set not containing x.

Since X is regular, the prover can find open sets U and V in X such that $x \in U$, $C \subseteq V$, and U and V are disjoint. Now, $U \cap A$ and $V \cap A$ are disjoint open subsets of A, with $x \in U \cap A$ and $C \subseteq V \cap A$. ☐

Note that we used *two* crucial facts. The first was that x being a point in the subspace, remained a point in the whole space. The second was that every closed set in the subspace was an intersection with the subspace of a closed set in the whole space.

The above proof works even without the accompanying T_1 assumption of regularity.

6.3. **Products of regular spaces.** In the proof that the product of two Hausdorff spaces is Hausdorff, the prover essentially identified the coordinate in which the two distinct points differed, and then separated the points via cylinders over open sets in that coordinate.

For the case of regular spaces, we are supplied with the datum of a point and a closed set. The point can still be expressed in terms of its coordinates. But the closed set can be expressed in terms of closed sets in the factor spaces only if it is a *closed box*.

For instance, suppose we look at the closed set in $X \times X$ (say, when X is regular and hence Hausdorff) given by the diagonal $D = \{x, x | x \in X\}$. How do we describe D in terms of its coordinates in X? The projection of D on the first coordinate gives the whole of X. Similarly, the projection of D on the second coordinate also gives the whole of X. However, all the points in $X \times X$ are not in D because the values of the two projections are not independent of each other.

Thus, we may not be able to find any coordinate where the point, and the projection of the closed set, are disjoint. For instance, if x is a point not in D in the above example, it is not true that x and the projection of D on either copy of X are disjoint. Moreover, there is another complication with this approach – the projection of a closed set in the product may not be closed.

However, we can use the alternative characterization of regularity in terms of a point and an *open* set containing it.

Claim. An arbitrary product of regular spaces is regular in the product topology.

Proof. Let $\{X_w\}_{w \in W}$ be a family of regular spaces and X be the product space. We must show that given a point $x \in X$ and an open set U containing x there is an open set V containing x such that $\overline{V} \subseteq U$.

Because U is an open set containing x, it must contain a basis open set containing x. The basis open sets in the product topology are cylinders about x. Take one such cylinder containing x, inside U. This looks like a product of proper open subsets of finitely many X_w with the whole spaces for the remaining X_w. Suppose X_1, X_2 until X_n are the spaces for which the open subsets are proper. Consider the projection on any of the X_i for i between 1 and n. The image of U (say U_i) is open under the projection, and the image of x (say x_i) lies inside that projection. Thus, by regularity of X_i, there is an open set V_i containing x_i such that $\overline{V_i} \subseteq U_i$.

Now, consider the open cylinder in X given as $V_1 \times V_2 \dots V_n$ times the full spaces for the remaining X_w. This is clearly an open set in the product topology. We now claim that if some point y lies in its closure, then y must be inside U. For, suppose $y \notin U$. Then, for some i, it is true that $y_i \notin U_i$, and hence, that $y_i \notin \overline{V_i}$. Thus, there is an open set in X_i that contains y_i but does not intersect V_i. Simply take the cylinder over this open set and we get an open set containing y but not V. Hence, $y \notin \overline{V}$ □

Concept Testers

(1) Prove the general statement that the closure of an open box is the corresponding closed box whose each projection is the closure of the corresponding open projection.
(2) Using the above statement, prove that the product of regular topological spaces is regular in the box topology.

Points to Ponder

- For the properties of T_0, T_1, T_2 we had observed that adding open sets makes the prover's job easier. However, in the case of regularity, it is not very obvious that this should be so. This is because the addition of more open sets not only gives the prover more options, it also gives the spoiler more options in selecting a point and an open set.

  Can you prove, or find a counterexample, to the assertion that given a regular topological space, shifting to a finer topology gives another regular topological space?

6.4. **Locally regular spaces.** Regularity is hereditary, that is, every subspace of a regular space is regular. There is a somewhat weaker notion called local regularity, similar to the notion of being locally Hausdorff. Here's the formal definition:

Definition. A topological space is said to be **locally regular**(defined) if every point has an open neighbourhood that is a regular space in the subspace topology.

Concept Testers

(1) Prove that a product of locally regular topological spaces is locally regular.
(2) A topological space is said to be **semiregular**(defined) if the regular open sets form a basis. Prove that any semiregular topological space is locally regular.

Points to Ponder

- What happens if we analogously define the notion of semi Hausdorff? Does it collapse and become equal to any of the properties already studied?

6.5. **What we've done so far.** We have now seen the properties T_0, T_1, T_2 and T_3. These are separation properties in increasing order of strictness. That is, we have:

$$T_3 \implies T_2 \implies T_1 \implies T_0$$

A topological space is termed T_3 if points are closed and any point and closed set not containing it can be separated by disjoint open sets. There's a little confusion as to what *regular* means. Some topologists use "regular" interchangeably with T_3. Others call a topological space regular if points and closed sets can be separated by disjoint open sets, regardless of whether or not points are closed.

For many purposes, it does not matter, because the spaces we usually consider have points closed anyway.

In the article, except for the subsection where we actually define regularity, we have used the term regular to mean T_3. However, many of the proofs we give actually work without the T_1 assumption.

Pause and Recollect

(1) Look at all the metaproperties we have established for regularity. Which of them are valid if we remove the T_1 assumption?

7. Normality

7.1. **Normal spaces.** The term *normal* is much abused in mathematics, and has different meanings in group theory, ring theory, geometry, topology, and in fact, it has multiple meanings within topology. The normal that we mean here is normal as a property of topological spaces. In this setup, the spoiler chooses *disjoint closed sets* and the prover chooses *disjoint open sets* containing them.

Definition. A topological space is said to be **normal**(defined) if given two disjoint closed sets in it, there are disjoint open sets containing the closed sets. That is, a topological space X is normal if, whenever A and B are closed sets in X with no intersection, there are open sets U and V with $A \subseteq U$, $B \subseteq V$ and $U \cap V$ empty.

A topological space is said to be $T_{4\text{(defined)}}$ if it is T_1 and normal.

Just as for regularity, we typically assume, in the case of normality, that the underlying topological space is at least T_1. In other words, we assume that points are closed. Under this assumption T_4 and normal get to mean the same thing.

With the expertise we have acquired in the spoiler-prover games, we can already ask basic questions about normality. These are explored below.

Concept Testers

(1) Prove that, under the T_1 assumption, all normal topological spaces are regular.
(2) Prove that every discrete space is normal.

Points to Ponder

- What problems would be there in carrying over the various proofs for Hausdorffness and regularity, to normality?
- For Hausdorff spaces, it was clear that the finer the topology, the easier the prover's job. However, for regular spaces, we found that both the spoiler and the prover get more options when the topology becomes finer. What happens in the case of normality?

7.2. **Metaproperties of normality.** The proof that every subspace of a Hausdorff space is Hausdorff was quite trivial – all the prover did was to separate the two points of the subspace by open sets in the whole space, and then intersected these open sets with the subspace. There was clearly no problem in going *from the subspace to the whole space* because distinct points in the subspace are distinct points in the whole space.

In the case of regularity, we had to argue a little more carefully.We started with a point and a closed set in the subspace, and then proceeded to the whole space. The closed set was expanded to a closed set in the whole space, and it was observed that this bigger closed set was also disjoint from the point.

However, in the case of normality, we have *two* closed sets in the subspace. When we expand both of them to closed sets in the whole space, the closed sets in the whole space might intersect each other somewhere outside the subspace. So, we cannot take the step of moving from the subspace to the space.

We can, however, prove the following:

Claim. Every closed subspace of a normal space is normal.

Proof. The idea is that because the subspace is already closed, closed subsets of it are already closed in the whole space. So we do not have to expand the closed subsets.

We now separate the closed subsets in the whole space. We get disjoint open sets of the whole space. Now, simply intersect these open sets with the subspace, to get disjoint open sets of the subspace separating the two disjoint closed sets. ☐

Let's also look at another proof that went through for T_0, T_1, T_2 and T_3 spaces: the property of being preserved under arbitrary products, in the box topology as well as in the product topology. For T_0, T_1 and T_2 we followed a uniform strategy of finding the differing coordinate and separating over that coordinate. For T_3, the same proof didn't work – we had to modify it a bit by looking at the open set containing the point, and finding a basis set inside that.

This kind of proof strategy just doesn't work for normal spaces. And not surprising: because the property of being normal is *not* preserved under products.

Points to Ponder

- A property of topological spaces is said to be **weakly hereditary**(defined) if every closed subspace of a topological space with the property also has the property. What we've shown is that the property of normality is weakly hereditary.

  Is the property of being collectionwise Hausdorff also weakly hereditary? If π is weakly hereditary, is locally π also weakly hereditary?

7.3. **Hereditarily normal spaces.** We begin with an obvious definition.

Definition. A topological space is said to be **hereditarily normal**(defined) if every subspace of it is normal in the subspace topology. Hereditarily normal topological spaces are also termed **completely normal**(defined). A topological space is said to be T_5(defined) if it is completely normal and T_1.

Once again, just as for normality, we assume the T_1 condition, so being completely normal is the same as being hereditarily normal.

We now give an alternative characterization of hereditarily normal spaces:

Claim. Two subsets in a topological spaces are said to be **separated subsets**(recalled) if neither intersects the closure of the other. A topological space is completely normal if and only if any pair of separated subsets can be separated by disjoint open subsets.

Proof. Suppose a given topological space X is completely normal. The spoiler picks two separated subsets A and B. The prover has to find disjoint open sets separating them.

The idea is to remove all the points of $\overline{A} \cap \overline{B}$. Notice that no point of A lies in $\overline{B}$, and no point of B lies in $\overline{A}$. So removing all these points gives a subspace containing both A and B. Moreover, the set of points removed is an intersection of two closed sets, and hence closed. So the subspace $Y = X \setminus \overline{A} \cap \overline{B}$ is open in X.

In this subspace, we claim that the closures of A and B are disjoint. This follows from the fact that we anyway removed al the common limit points. Once these closures are disjoint sets, they can be separated by disjoint open sets in Y. Because Y itself is open in X, these disjoint open sets in Y are in fact open in X, and the prover has got hold of disjoint open sets.

Now let's prove the converse. Suppose the prover can always separate a pair of separated sets with disjoint open sets. He now has to show that the space is completely normal.

The spoiler picks a subspace Z of X, and two closed sets A and B in that subspace. He then asks the prover to separate them by disjoint open sets in the subspace. The prover simply observes that the closure of A in X must intersect Z in precisely A, and hence, cannot contain points of B. Similarly, the closure of B cannot contain any points in A. Thus, A and B are separated in X, and there are disjoint open sets in X separating them. Intersecting these with Z, the prover obtains disjoint open sets of Z separating A and B. □

Concept Testers

(1) Prove that if, for a given topological space, every open subspace is normal, then the space is hereditarily normal. (If you get stuck, just go through the proof of the above claim carefully).

Points to Ponder

- Suppose a property π of topological spaces is hereditary, that is, every subspace of a space with property π has property *pi*. Prove that if π is stronger than normality, then π is in fact stronger than the property of being completely normal.

  Can you think of some candidates for the property π?

7.4. **Product of normal spaces.** The niceness of the low levels of separation is rapidly lost as we go higher up. In particular, it is no longer true that a product of normal spaces is normal. Of course, the proof that we gave for Hausdorffness cannot work – in fact, it did not work even for regularity. Nor can we adapt the proof for regularity to normality.

7.5. **Pseudonormal spaces.** Within the assumption of the T_1 axiom, we see that normality is at one extreme: *any* two disjoint closed sets can be separated by disjoint open sets. hat is, the spoiler is given complete freedom in choosing the disjoint closed sets.

Regularity, on the other hand, requires the spoiler to ensure that one of the closed sets he picks is a singleton set. That is, the spoiler's freedom is considerably restricted. Thus, the property of regularity is not such a strong one, because the prover's job is somewhat easier.

We are interested in understanding the gap between normality and regularity. That is, we would like to restrict at least one of the sets the spoiler chooses in a suitable way, without being so strict as to allow him only to pick a point.

One such in between property is (again, assuming the T_1 condition):

Definition. A topological space is said to be **pseudonormal**(defined) if given two disjoint closed sets, one of which is countable (or finite), there are disjoint open sets separating them.

7.6. **A quick summary.** So far, the primary separation properties we have seen are:

Property	Spoiler picks	Prover finds
T_0	2 distinct points	Open set containing one, not the other
T_1	2 distinct points, one marked	Open set containing marked, not other
T_2	2 distinct points	Disjoint open sets containing them
Collectionwise T_2	Discrete set of points	Pairwise disjoint open sets containing them
Regular	One point, one closed set	Disjoint open sets containing them
Normal	Two disjoint closed sets	Disjoint open sets containing them
Completely normal	Separated sets	Disjoint open sets containing them
Pseudonormal	Disjoint closed sets, one countable	Disjoint open sets containing them

In all of these except T_0 and T_1 the prover has to pick *disjoint* open sets. However, in the separation setup that we had described earlier on in section 3.1, we had talked of many other notions of separation. In particular, we had talked of *separated*, *closure disjoint* and *function separated*. In the next section, we will study analogues of the separation axioms seen so far where the spoiler's move remains the same, but the prover has to locate, not just disjoint open sets containing them, but also ensure they are separated, closure disjoint, or function separated.

8. Notable results

8.1. Constructing continuous functions.

What is a continuous function from a topological space to $[0,1]$? Essentially, we must ensure that the inverse image of every open interval (relative to $[0,1]$) remains open. Note that $[0,a)$ and $(a,1]$ are open sets in $[0,1]$. Clearly, thus, the inverse image of $[0,a)$ is open, and the inverse image of $(a,1]$ is open. The complement of the union of these two open sets is the inverse image of the point a, which is a closed set.

Now, suppose $f : X \to [0,1]$ is continuous, and $f(A) = 0$ and $f(B) = 1$ for subsets A and B of X. Clearly, $A \subseteq f^{-1}[0,1/2)$ and $B \subseteq f^{-1}(1/2,1]$ giving that A and B are contained in disjoint open sets. We can, in fact, do better. We have $A \subseteq f^{-1}[0,1/4) \subseteq f^{-1}[0,1/4]$, and $B \subseteq f^{-1}(3/4,1] \subseteq f^{-1}[3/4,1]$. Clearly, the set $f^{-1}[0,1/4)$ is an open set containing A and its closure is contained in $f^{-1}[0,1/4]$. Similarly, the set $f^{-1}(3/4,1]$ is an open set containing B and its closure is contained in $f^{-1}[3/4,1)$. Thus A and B are contained in two closure disjoint open sets.

We can, in fact, repeat this process indefinitely. Having found two closure disjoint sets, we can again separate their closures by disjoint open sets. Here, the closed sets $f^{-1}[0,1/4]$ and $f^{-1}[3/4,1]$ can again be separated by, say, $f^{-1}[0,3/8)$ and $f^{-1}(5/8,1]$ which are again closure disjoint. Taking their closures, we can again separate by closure disjoint open sets, and so on.

In essence, using the continuous function, we have obtained a hierarchical separation into open sets that are closure disjoint.

The interesting question is: under what circumstances can we reverse the procedure? That is, if we know how to construct separating open sets, actually obtain a function from the space to $[0,1]$ that separates two given sets? A strong result in this direction is the Urysohn lemma. Before discussing the result, however, we'll take a quick diversion.

8.2. Bisection Search.

How do we compute $\sqrt{2}$? We compute it digit by digit. When computing the first digit, we show that $\sqrt{2}$ lies in $[1,2)$. Then, when we show it is 1.4... we show that it lies in the open interval $[1.4,1.5)$. When we show it begins with 1.41 we have shown that it is in $[1.41,1.42)$. That is, finding the *value* essentially means keeping on getting smaller and smaller open sets in which the value lies.

Because we work in the decimal system, the size of the open interval which represents our **interval of uncertainty**(explained) reduces each time by a factor of 10. If we were working in 2, the size of the set would reduce each time by a factor of 2.

Suppose we wanted to compute a number which we know is between 0 and 1. All we can do is give an open interval and ask the question: is the number in that open interval? The natural approach may be as follows: first, look at $(0,1)$. If the number is not in the interval, it must be either 0 or 1. If it is in $(0,1)$, then we consider $(0,1/2)$ and $(1/2,1)$. If it lies in $(0,1/2)$ we search inside that half, and if it lies in $(1/2,1)$ we search inside that half. If it lies in neither, it must be at $1/2$.

In the first stage, we narrow down the size of the **search space**(explained) by a factor of two. In fact, at every iteration, it narrows the search space by half. As the size of the search space shrinks to zero, the open sets ultimately converge to a unique value, which is the value we were searching for.

In popular jargon, this (or rather, a slight variant of it) is termed **bisection search**(defined)[1]

What Urysohn did with bisection search was to turn it around and use it as a technique for defining a function. The idea is the following: we *define* the number based on the results of the bisection search, rather than first defining the number and then determining it via bisection search. This is the so called **bisection search recipe**(explained)

> We define a rule that, for a certain basis of open sets (all intervals) for $[0,1]$, returns either true or false, such that:
> - If one basis member is inside the other, the function returns true
> - If it returns true on an basis member, there is an basis member of half the size on which it also returns true
> - If it returns true on an basis member, then it returns true on every basis member containing it.
>
> This rule pinpoints a unique number in $[0,1]$.

8.3. **Urysohn lemma.** Urysohn's lemma is one of the first "nontrivial" pieces of topology. That is, it is one of the first few results whose proof method does not suggest itself from the statement. However, we have already built the foundation for the proof in the last two subsections.

Theorem 1 (Urysohn Lemma)**.** Given two closed subsets A and B of a normal space X, there is a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

To *describe* the function that separates the two closed subsets, we'll create a bisection search recipe. That is, our construction is aimed so that for each point x, we have a bisection search recipe to locate $f(x)$. We'll further claim that the function we get via the bisection search recipe is continuous, and that $f(A)$ and $f(B)$ are at the two extremes.

Proof. First, by normality, separate A and B by disjoint open sets U_1 and U_2. The complement of the union of these is closed. Send all points in this complement (say C) to $1/2$. Our aim is to send the whole of U_1 to $[0,1/2)$ and the whole of U_2 to $(1/2,1]$. We need to, however, decide how to do this.

Separate A and C by disjoint open sets, say V_1 and V_2. The idea is to send V_1 to $[0,1/4)$ and V_2 to $(1/4,1/2]$. So, we first look at those points in U_1 which are in neither V_1 nor V_2. Send these points to $1/4$.

Similarly, separate C and B by disjoint open sets, say W_1 and W_2. The idea is: send W_1 to $[1/2,3/4)$ and W_2 to $(3/4,1]$. Again, we cannot exactly allocate values to the points in W_1 or W_2. What we do instead is: send points of U_2 that are in neither W_1 nor in W_2 to $3/4$.

We now have four parts, V_1 (which we plan to send somewhere in $[0,1/4)$, V_2 (which we plan to send somewhere in $(1/4,1/2]$ W_1 (which we plan to send in to $[1/2,3/4)$ and W_2 (which we plan to send in to $(3/4,1]$. And, we have identified closed sets mapping to the points 0, 1/4, 1/2, 3/4 and 1. That is, we have a collection of open intervals at whose endpoints we have a closed inverse image known.

For *each* of these, separate the bounding closed sets by disjoint open sets, and send whatever part is in neither open set to the midpoint of the interval. Every time we do this, the size of the intervals becomes half. Thus, as we keep doing this, we ultimately end up with assigning values of the form $a/2^n$ for many points.

The computation of $f(x)$ for any point $x \in X$ now boils down to a bisection search recipe. For each x, first check if it is in U_1 or U_2. If it is in U_1, we have narrowed $f(x)$ to $[0,1/2)$. if it is in ☐

I'll reiterate the main steps of the proof:

- We start off with two closed sets.
- We separate them by two open sets – the left half, and the right half. The *in betweens* map to 1/2.
- We now partition each half into two halves. To do this partitioning, we apply normality, this time taking our two open sets at the end point and the middle point respectively. Again, the points which go in neither open set get sent to the midpoint of the corresponding interval.
- We proceed this way, indefinitely.
- To assign the value to a point we perform a bisection search.

[1]In bisection search, we may not be looking for a specific value, but rather, searching for any one point from a given set. Then, each time we query with an open interval, we are told if there is at least one point from the set in the interval. For instance, finding the zeroes of a continuous function that changes sign across an interval.

8.4. **Intermediate notions of separation.** The next question of natural interest is: what aspect of the Urysohn lemma fails when we handle regular spaces? That is, can we use the same reasoning to show that in a regular space, a point and a closed set can be separated by a continuous function? If not, why?

The problem is this. Recall that the proof for normality went something like this: start by separating the two closed sets by disjoint open sets. Think of the first set as being the *left half*, that is, the inverse image of $[0, 1/2)$, and the second set as being the inverse image of the *right half*, that is, $(1/2, 1]$. And whatever is left over goes to $1/2$. Then we separated the inverse images of 0 and $1/2$ by disjoint open sets, and we similarly separated the inverse images of $1/2$ and 1 by disjoint open sets.

In case we begin by assuming regularity, then we can do the first step. Separate the point and the closed set by disjoint open sets. Think of the open set containing the point as $[0, 1/2)$ and the open set containing the closed set as $(1/2, 1]$. The inverse image of $1/2$ is again a closed set. We can now, using regularity, separate 0 and $1/2$ by disjoint open sets. However, to separate $1/2$ and 1, we need to separate two *closed* sets, and regularity does not provide for this.

The question: is normality *necessary* to be able to separate points and closed sets by continuous functions? Or are there other kinds of properties that we could potentially use? The typical other kind of example is what is called a **uniform structure**(first used). This is some kind of structure that enables us to copy behaviour from one point to another. Thus, after separating the point and the closed set by disjoint open sets, we again try to separate the point and the complement of their union. Thus, we can keep doing the construction through one side, from $1/2$ to $1/4$ and so on. The problem is in doing it at the other side. Uniformity helps us here.

This leads to the following definition:

Definition. A topological space is said to be **completely regular**(defined) if given a point and a closed set, there is a continuous function from the space to $[0, 1]$ taking the value 0 at the point and 1 on the closed set.

As a summary of what we have discussed above, we can say the following. The spoiler picks two things to be separated. The prover can then:

- Separate them by disjoint open sets
- Separate them by closure disjoint open sets
- Separate them by a continuous function

We now have:

- When the spoiler is allowed to pick two disjoint closed sets, all the three above notions give rise to the same topological space property: the property of *normality*.
- When the spoiler is allowed to pick a closed set and a point not in it, the first two notions become equivalent, but the third notion is different. The first two notions correspond to the property of *regularity* and the third to the property of *complete regularity*.
- When the spoiler is allowed to pick two distinct points, all three notions are different. The first is the property of being *Hausdorff*, the second is the property of being **completely Hausdorff**(defined) and the third is the property of being **Urysohn**(defined).

8.5. **A notion of limit of functions.** Earlier on in this journey, we explored the notion of limit of a sequence of points. Another important related notion is that of limit of a sequence of functions. Here is one definition:

Definition. A sequence of functions $f_n : X \to Y$ as $n \in \mathbb{N}$, is said to be **pointwise convergent**(defined) (or is said to converge pointwise) to a function $f : X \to Y$ if for every $x \in X$, the sequence $f_n(x)$ tends to $f(x)$.

The problem with this definition of convergence (as we will see in detail in a later journey) is that given a sequence of continuous functions that converges pointwise to a given function, the given function may not itself be continuous.

So, what notion of limit must be used so that a sequence of continuous functions, if convergent, converges to a continuous function? We'll here define the notion for functions to $\mathbb{R}$, though the same definition applies without problem to an arbitrary metric space.

Definition. Let $f_n : X \to \mathbb{R}$ be a sequence of functions and $f : X \to \mathbb{R}$ be a function. Then, f_n is said to be **convergent**(defined)uniformly to f if for any $\epsilon > 0$, all but finitely many f_n satisfy $|f_n(x) - f(x)| < \epsilon$ for every $x \in X$.

Essentially, uniform convergence means that a sequence of functions converges to a given function simultaneously at all points, at a fast rate. It cannot happen that it converges on some part very fast and takes arbitrarily long to converge at another point. To describe this notion of "fast at all points" we need to have a way of comparing the rate of convergence at various points. That is, we need ways of making statements like: x is closer to y than z is to w.

Such statements cannot be made from the purely topological viewpoint. We need a structure that can keep track of and compare *relative distances*. One way to give this structure is using a metric or distance function. There are other ways as well. We'll explore this issue in another journey.

For the purpose of this journey, the important result is as follows:

Claim. If a sequence of continuous functions $f_n : X \to \mathbb{R}$ is uniformly convergent to $f : X \to \mathbb{R}$, f is a continuous function.

I will not give a proof here, because the picture is going to become much simpler after we study metric spaces. The proof is a modification of the *three part triangle inequality* idea used for proving the analogous result for functions $\mathbb{R} \to \mathbb{R}$.

Concept Testers

(1) Define $f_n : \mathbb{R} \to \mathbb{R}$ as follows. $f_n(x) = \max\{0, 1/n - |x|\}$. Prove that f_n is pointwise convergent but not uniformly convergent.

8.6. **Tietze extension theorem.** The Tietze extension theorem is perhaps an excellent example of a fairly tricky and clever proof using the Urysohn lemma. Before stating the theorem and its proof, let me build a little by way of motivation.

Suppose A is a subset of B. Then, any function from B to a set S naturally gives a function from A to S via the so called "restriction" operation, simply map each element of A to whatever it was getting mapped to when an element of B. The question is: if we start with a function from A to S, how do we find a function from B to S, whose restriction to A gives the function we have? Obviously if we are just looking at set theoretic functions, we can send the remaining elements of B to any element in S and we get an extension. However, when we require our functions to satisfy some additional conditions, then the question of lifting becomes a fairly intricate one.

For instance, we know that the continuous function $x \mapsto 1/x$ on the set $\mathbb{R}^*$ (that is, nonzero reals) cannot be extended to a continuous function on the whole of $\mathbb{R}$. On the other hand, the function $x \sin(1/x)$ can be extended to 0, even though it is not naturally defined at 0. The function $x \mapsto e^x$ defined on $\mathbb{R}$ can be extended to the function $z \mapsto e^z$ on the entire complex plane. The function $x \mapsto \log x$ on the nonnegative reals can be extended to a continuous function on $\mathbb{C}$ minus a so called "branch", but cannot be extended to a continuous function on all the nonzero complex numbers.

With this background, we state the Tietze Extension Theorem.

Theorem 2 (Tietze Extension Theorem)**.** Any continuous function from a closed subset of a normal topological space to $[0, 1]$ can be extended to a continuous function from the whole space to $[0, 1]$.

Let X be the normal space, A the closed subspace, and f the continuous map from A to $[0, 1]$. We now construct g on X as an infinite summation, in such a way that for all points in A, the summation equals the value of f.

The idea is to to look at the inverse images of *extreme intervals* in $[0, 1]$, that is, things like $[0, a]$ and $[b, 1]$ with $a < b$. The inverse image of such an interval via f is closed in A, and because A is closed in X, it is also closed in X. Thus, the inverse images of $[0, a]$ and $[b, 1]$ are disjoint closed sets in X. By the Urysohn lemma, we can separate them by a continuous function.

We seek to keep locating pairs of disjoint closed sets and getting more and more continuous functions so that the sum converges to the value attained by f. To make the proof work, we must choose our intervals somewhat carefully. That is what the formal proof below is about.

Proof. For reasons of ease and symmetry, we work with the interval $[-1, 1]$ instead of the interval $[0, 1]$. There's no problem switching between the two because the interval $[-1, 1]$ and the interval $[0, 1]$ are homeomorphic – in fact $[-1, 1]$ can be thought of as $[0, 1]$ dilated about 1 by a factor of 2.

So we have $f : A \to [-1, 1]$. We now divide $[-1, 1]$ into three equal slices, getting the cutting points as $-1/3$ and $1/3$. Let $C_1 = f^{-1}[-1, -1/3]$ and $C_2 = f^{-1}[1/3, 1]$. Then, $f^{-1}(C_1)$ and $f^{-1}(C_2)$ are disjoint closed subsets of A. Because A is closed in X, they are in fact disjoint closed subsets of X.

We now find a function, say g_1, that separates them. However, instead of looking at the image of g_1 as $[0,1]$, we take it as $[-1/3, 1/3]$. Again, we can do this because any two closed intervals can go to one another via translations and dilations. Observe now that for each slice, $|f(x) - f_1(x)| < 2/3$:

- When $f(x) \leq -1/3$, $f_1(x) = -1/3$, so the distance between $f(x)$ and $f_1(x)$ can be at most $2/3$.
- When $f(x)$ lies between $-1/3$ and $1/3$, so does $f_1(x)$, so the distance between them can again be at most $2/3$.
- When $f(x) \geq 1/3$, $f_1(x)$ is $1/3$, so the distance is at most $2/3$.

Notice that we have made *progress* in the following sense:

- We now have a function f_1 on the *whole* of the space.
- $f(x)$ and $f_1(x)$ differ by at most $2/3$ on the whole space.

We now consider the function $g(x) = f(x) - f_1(x)$ on A. This is a function from A to $[-2/3, 2/3]$. Thus, $x \mapsto 3g(x)/2$ is a function from A to $[-1, 1]$. So, we can find another function, say h, such that $|3g(x)/2 - h(x)| < 2/3$ for points in A. Put $f_2(x) = f_1(x) + 2h(x)/3$.

Now consider $|f(x) - f_2(x)|$on A. By construction, we know that $|3g(x)/2 - h(x)| < 2/3$, so $|f(x) - (f_1(x) + f_2(x))| < (2/3)^2$. In other words, $f_2(x)$ is a *better* approximation – the error is only $(2/3)^2$.

Again, take $g_2(x) = f(x) - f_2(x)$. This is a function from the whole space to $[-4/9, 4/9]$. Again take $(3/2)^2 g_2(x)$ and find a corresponding $h(x)$. This gives a new function f_3 such that $|f(x) - f_3(x)| < (2/3)^3$.

Applying the construction inductively, we obtain a sequence f_n of functions defined on the whole set X that converge to f for all points which are inside the closed set A. Thus, on A, the *pointwise* limit of f_n is the same as the function f.

It remains to show that the pointwise limit of f_n is actually a continuous function. This follows essentially by a Cauchy criterion for convergence. We know that for any $\epsilon > 0$, there is an N such that for all $n \geq N$ and $x \in X$, $|f_{n+1}(x) - f_n(x)| < \epsilon$. Thus, f_n is pointwise convergent, and converges uniformly to the pointwise limit. A uniform limit of continuous functions is continuous, and we thus get $\lim_{n\to\infty} f_n$ is a continuous extension of f to X. □

Points to Ponder

- A topological space Y is said to have the **Universal Extension Property**(defined) if given any topological space X, a closed subspace A of X, and a continuous function f from A to Y, there is a lift of f to a continuous function from X to Y. The Tietze extension theorem is essentially a proof that $[0,1]$ has the Universal Extension Property. Does the unit square have the Universal Extension Property?

8.7. **A summary.** The Urysohn Lemma and the Tietze extension theorem are very important results both in terms of the statement of the result and the proof details. The significance of results such as the Tietze extension theorem lies far outside the realm of separation properties. The Urysohn lemma result (and subsequent observations) can be summarized as below:

Spoiler picks	Open set sep.	Closure disjoint sep.	Function sep.
Two points	T_2	Completely T_2	Urysohn
Point, closed set	Regular	Regular	Completely regular
Two closed sets	Normal	Normal	Normal

9. What we achieved in this journey

9.1. **The explicit promise.** At the outset of the journey, we had planed to do the following:

- Define the following: T_0, T_1, T_2 (Hausdorff), T_3 (regular), T_4 (normal)
- Understand how these properties are affected on taking subspaces, products and other similar constructions
- Prove the Urysohn Lemma and the Tietze Extension Theorem

It is clear that we achieved all these aims. The properties of T_0 and T_1 are equal to their local variants. The properties of T_0, T_1, T_2 and T_3 are all closed under going to subspaces and taking products. This no longer holds for the property of normality.

The powerful Urysohn Lemma tells us that the ability to separate disjoint closed sets by disjoint open sets guarantees the ability to separate disjoint closed sets by a continuous function.

Finally, with the Tietze extension theorem, we considered the problem of extending a continuous function from a closed set to the whole space, and found that it was always possible when the function was to $[0,1]$. That is, a function *from* a *closed* subset of a *normal* space to $[0,1]$ can be extended in *at* *least one* way. Note the contrast with another result we encountered in the journey: the statement that a function *to* a Hausdorff space defined on a *dense* subset of a space can be extended in *at most one* way.

9.2. **Separation: axioms or properties?** The use of the word "separation axioms" for separation properties of topological spaces is not accidental. Depending on what kind of topological spaces a mathematician typically dealt with, he had a corresponding notion of "decency" for topological spaces. Mathematicians working primarily with metric spaces concern themselves only with normal spaces – probably one of the reasons why such spaces have been called "normal" (as in typical, ordinary). More adventurous mathematicians working with somewhat less nice structures want at least some kind of regularity in the structure – and they came up with the notion of *regular*.

The lower separation properties (T_2, T_1 and T_0) are a source of confusion. Some mathematicians include the T_2 assumption as an axiom of topological spaces – for them "topological space" is synonymous with "Hausdorff topological space". The assumption of T_1 is a fairly mild one which practically all spaces of direct interest satisfy. The T_0 assumption has recently come to generate more interest, thanks to its use in modelling some structures in computer science that fail to be T_1.

Going below T_0 is not necessary because of the so called "Kolmogorov quotient construction". We'll discuss this construction in a later journey.

9.3. **How we use these.** A clear idea of the separation properties typical of the spaces we are studying helps us understand what kind of proof techniques to use. For instance we know that:

- When working with T_1 spaces, we use that points are closed. In particularly, the inverse image (via a continuous function) of a point in a T_1 space is a closed set.
- When working with T_2 spaces, we show *uniqueness* of a point by separating two distinct points by disjoint open sets.
- When working with T_3 spaces, we can get a layered separation between a point and a closed set.
- When working with T_4 we can separate disjoint closed sets by continuous functions.

Every separation property gives different *proof techniques.*

It is also important to keep track of whether, by taking subspaces and products, we might run the risk of going *outside* of the separation properties. The stronger the closure properties we can ensure, the better things are. We'll keep coming back to this theme in later journeys.

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