LECTURES 3 AND 4 OF LIE-THEORETIC METHODS

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Abstract. These are lecture notes for the course titled Lie-theoretic methods in analysis taught by Professor Alladi Sitaram. In these lecture notes, Professor Sitaram defines a linear Lie group and the exponential map and discusses preliminary results on these.

1. Linear Lie groups

1.1. What is a linear Lie group?

Definition. A linear Lie group (defined) is a group along with an embedding as a closed subgroup of $GL(n, \mathbb{R})$.

Some comments:
1. $GL(n, \mathbb{C})$ can be embedded as a closed subgroup of $GL(2n, \mathbb{R})$ by treating $\mathbb{C}$ as a two-dimensional vector space over $\mathbb{R}$.
2. $GL(n, \mathbb{R})$ is an open subset in $\mathbb{R}^{n^2}$ and the multiplication and inverse maps are smooth maps.
3. $GL(n, \mathbb{R})$ is not connected. It has two connected components: the subgroup comprising matrices of positive determinant (denoted as $GL^+(n, \mathbb{R})$) and the subgroup comprising matrices of negative determinant.

1.2. Examples of linear Lie groups. The following are linear Lie groups, either directly viewed as closed subgroups of $GL(n, \mathbb{R})$, or viewed as closed subgroups of $GL(n, \mathbb{C})$ (and hence, by transitivity of closedness, as closed subgroups of $GL(2n, \mathbb{C})$).

1. $GL(n, \mathbb{R})$ itself
2. $GL^+(n, \mathbb{R})$
3. $SL(n, \mathbb{R})$ is the group of matrices with positive determinant
4. $O(n, \mathbb{R})$ is the group of matrices $A$ such that $AA^T$ is the identity matrix
5. $SO(n, \mathbb{R})$ is the intersection of $O(n, \mathbb{R})$ and $SL(n, \mathbb{R})$. Note that $SO(n, \mathbb{R})$ is the connected component of $O(n, \mathbb{R})$.
6. $U(n, \mathbb{C})$ is the group of matrices $A$ over $\mathbb{C}$ such that $AA^*$ is the identity matrix.

1.3. Proofs of connectedness. We now show how the problem of proving connectedness of $GL^+(n, \mathbb{R})$ can be reduced to the problem of connectedness of $SO(n, \mathbb{R})$.

We make a series of observations:
1. Every invertible matrix can be expressed as a product of an orthogonal matrix and a positive definite matrix. In particular, every invertible matrix of positive determinant can be expressed as a product of a special orthogonal matrix and a positive definite matrix.
2. Every positive definite matrix can be expressed as the exponential of a symmetric matrix. This follows by first diagonalizing (using an orthogonal matrix) and then writing the corresponding diagonal matrix as the exponential of a diagonal matrix. Since conjugating by an orthogonal matrix preserves the property of being symmetric, the conclusion holds.
3. The upshot of these is that the map:

$$\text{Orthogonal matrices } \times \text{Symmetric matrices } \rightarrow GL(n, \mathbb{R})$$
given by $(X, Y) \mapsto X \exp(Y)$ is surjective. It is in fact a set-theoretic bijection.

In particular, if we restrict to special orthogonal matrices, we get all elements of $GL^+(n, \mathbb{R})$.
The above map is continuous. Hence, if we are able to show that $SO(n, \mathbb{R})$ is connected, we can also show that $GL^+(n, \mathbb{R})$ is connected.

The proof that $SO(n, \mathbb{R})$ is connected follows from the fact that every element of $SO(n, \mathbb{R})$ can be written in block form with each block looking like

$$
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
$$

Since each of these blocks can be obtained from the identity by moving the parameter $\theta$ from 0 to 1, the given matrix is connected to the identity. Since any conjugate of an element connected to the identity is also connected to the identity, every element is in the connected component of the identity.

In the complex numbers, every invertible matrix is a product of a unitary matrix and a positive definite matrix (for a Hermitian inner product).

Thus, a similar argument to the previous one shows that $GL(n, \mathbb{C})$ is connected.

1.4. Lie groups and linear Lie groups. The abstract notion of Lie group (which we shall not dwell upon here) is as a differential manifold with a group structure that is “compatible” in the sense that the multiplication and inversion maps are smooth.

Linear Lie groups are thus very special cases of Lie groups. The natural question is: under what conditions can an abstract Lie group be treated as a linear Lie group?

**Fact 1.** Every compact Lie group can be treated as a linear Lie group. Thus, as far as compact Lie groups are concerned, there is no loss of generality in assuming linearity.

1.5. The exponential map. We had seen the exponential map earlier. Let us now consider it formally:

$$\exp : M(n) \to GL(n)$$

is defined as follows:

$$\exp(X) = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} \ldots$$

Some observations:

1. The series for $\exp(X)$ is absolutely convergent, so there is no problem about rearranging the terms when manipulating the exponential series
2. $\exp(X)\exp(-X) = 1$ follows by just considering the series product and observing that the manipulations we do for real numbers are equally valid here because the series is absolutely convergent. Hence, in particular, $\exp(X)$ is in $GL(n, \mathbb{R})$.
3. $\exp(A + B) = \exp(A)\exp(B)$ whenever $A$ and $B$ commute (multiplicatively).
4. The Jacobian of the exponential map at the origin is the identity $n^2 \times n^2$ matrix.

From the Inverse Function Theorem, we now have:

**Fact 2.** There exists a neighbourhood $U_1$ of the origin in $M(n)$ and $U_2$ of the identity matrix in $GL(n)$ such that the exponential map, restricted to $U_1$, is a smooth ($C^\infty$) diffeomorphism from $U_1$ to $U_2$.

**Definition** (Lie algebra of a Lie group). Let $G$ be a linear Lie group. The Lie algebra of $G$, denoted by $\mathfrak{g}$, is defined as the subset of $M(n)$ comprising elements $X$ with the property that $\exp(tX)$ is in $G$ for every $t \in \mathbb{R}$.

It is clear that $\exp(tX)$ commute with each other, and in fact they form a one-parameter group inside $GL(n)$ (a homomorphic image of $\mathbb{R}$). This can also be viewed as a curve in $GL(n)$, in which case $X$ can be viewed as a tangent direction to the curve.
**Theorem 1** (Lie algebra is a Lie algebra). If $G$ is a linear Lie group and $\mathfrak{g}$ is its Lie algebra, then:

1. $\mathfrak{g}$ is a vector space over $\mathbb{R}$
2. Whenever $x, y \in \mathfrak{g}$, $xy - yx \in \mathfrak{g}$
   Combining the above two statements, $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$
3. There is a neighbourhood $U_1$ of 0 in $\mathfrak{g}$ and $U_2$ of the identity in $G$, such that the exponential map defines a diffeomorphism from $U_1$ to $U_2$. In fact, $U_1$ and $U_2$ can be taken as the intersections with $\mathfrak{g}$ and $G$ of the neighbourhoods corresponding to $M(n)$ and $GL(n)$ respectively.

We shall prove this theorem later. The significance of this result is that it makes $G$ into a differential manifold where the exponential map serves as a coordinate chart.

**Definition** (Dimension of a Lie group). The dimension (defined) of a linear Lie group is defined as the real dimension of its Lie algebra. Note that this is the same as its dimension as a differential manifold, because the Lie algebra is the vector space containing coordinate charts for the Lie group.

2. **More on the exponential map**

2.1. **Simple properties of the exponential map.** This includes properties we have already seen and properties we will see for the first time:

1. $\exp(0) = I$
2. $\exp(-A) = (\exp(A))^{-1}$
3. If $A$ and $B$ commute, $\exp(A + B) = \exp(A) \exp(B)$
4. $\exp(A^t) = \exp(A)^t$
5. $\exp(PAP^{-1}) = P\exp(A)P^{-1}$, that is, the exponential commutes with inner automorphisms
6. If $\lambda_i$ are the eigenvalues for $A$, $e^{\lambda_i}$ are the eigenvalues for $\exp(A)$.
7. If $A$ is upper triangular, so is $\exp(A)$
8. $\log$ is defined in a suitable neighbourhood of the identity, by:
   $$\log A = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(A - I)^n}{n}$$
9. In suitable neighbourhoods of 0 and $I$, $\exp$ and $\log$ are inverses of each other

2.2. **Harder properties of the exponential map.** These are properties that describe what happens when $A$ and $B$ are noncommuting matrices. Roughly the extent to which $A$ and $B$ fail to commute is correlated with the extent to which $\exp(A + B) = \exp(A) \exp(B)$ fails to hold.
   The three facts, in continuous form, look like:

1. $\exp(tX) \exp(tY) = \exp \left( t(X + Y) + \frac{t^2}{2} [X, Y] + O(t^3) \right)$
2. $\exp(tX) \exp(tY) \exp(-tX) = \exp \left( tY + \frac{t^2}{2} [X, Y] + O(t^3) \right)$
3. $\{\exp(tX), \exp(tY)\} = \exp \left( \frac{t^2}{2} [X, Y] + O(t^3) \right)$

   Here the curly braces denote commutator in the multiplicative group $GL(n)$

   The discretized versions (that are often useful for taking discretized limits):

1. $\lim_{n \to \infty} \left( \exp \left( \frac{X}{n} \right) \exp \left( \frac{Y}{n} \right) \right)^n = \exp(X + Y)$
2. $\lim_{n \to \infty} \left\{ \exp \left( \frac{X}{n} \right) \exp \left( \frac{Y}{n} \right) \right\}^n = \exp[X, Y]$
Curly braces denote the multiplicative commutator. Though the same results are true for abstract Lie groups (for suitably modified definitions of the exponential map) the results are much harder to prove there because we do not have explicit expressions paralleling the power series in the linear case.

2.3. The Lie algebra is a Lie algebra. In the previous section (lecture) we had stated that the Lie algebra of any Lie group is a Lie algebra, and further, that the exponential map defines a local diffeomorphism. Here, we prove that statement.

**Claim.** If $G$ is a linear Lie group and $\mathfrak{g}$ is its Lie algebra, $\mathfrak{g}$ is a vector space over $\mathbb{R}$.

**Proof.** To prove that $\mathfrak{g}$ is a vector space, we need to prove two facts:

- Given $X, Y \in \mathfrak{g}$, $X + Y \in \mathfrak{g}$.
- Given $X \in \mathfrak{g}$ and $\lambda \in \mathbb{R}$, $\lambda X \in \mathfrak{g}$.

The first of these is the more challenging. Essentially, we need to show that if $\exp(tX) \in G$ for all $t \in \mathbb{R}$ and $\exp(tY) \in G$ for all $t \in \mathbb{R}$, then $\exp(t(X + Y)) \in G$ for all $t \in \mathbb{R}$.

We use the limit form of the identity mentioned earlier:

$$\lim_{n \to \infty} \left( \frac{X}{n} \exp \left( \frac{Y}{n} \right) \right)^n = \exp(X + Y)$$

Replacing $X$ by $tX$ and $Y$ by $tY$, we obtain:

$$\lim_{n \to \infty} \left( \frac{tX}{n} \exp \left( \frac{tY}{n} \right) \right)^n = \exp(t(X + Y))$$

Note that for all $n$, since $\exp(tX/n)$ lies inside $G$ and $\exp(tY/n)$ lies inside $G$, the product also lies inside $G$. Hence, the left-hand side is the limit of a sequence of elements all inside $G$. Since $G$ is a closed subgroup of $GL_n$, the limit also lies inside $G$. Hence $\exp(t(X + Y))$ lies inside $G$.

The fact that $X \in \mathfrak{g} \implies \lambda X \in \mathfrak{g}$ follows directly from the definition of Lie algebra. □

**Claim.** If $X, Y \in \mathfrak{g}$, then the commutator $[X, Y] = XY - YX$ also belongs to $G$.

**Proof.** It suffices to show that if $\exp(tX), \exp(tY) \in G$ for all reals $t$, then $\exp(t[X, Y]) \in G$ for all real $t$.

We use the limit identity:

$$\lim_{n \to \infty} \left\{ \exp \left( \frac{tX}{n} \right) \exp \left( \frac{tY}{n} \right) \right\}^n = \exp([X, Y])$$

Here the curly braces indicate the multiplicative commutator.

Replacing $X$ by $tX$ and $Y$ by $tY$, we obtain:

$$\lim_{n \to \infty} \left\{ \exp \left( \frac{tX}{n} \right) \exp \left( \frac{tY}{n} \right) \right\}^n = \exp([tX, tY])$$

We find that the left-hand-side is the limit of a sequence of elements from $G$. Again, since $G$ is a closed subgroup, the left-hand-side is an element of $G$, and hence so is the right-hand-side. Thus, for every $t$, the element $\exp([tX, tY])$ is in $G$. Since $[tX, tY] = t[X, Y]$, we obtain that $\exp(t[X, Y]) \in G$ for all $t$, and we are done. □

2.4. More on the local diffeomorphism. Let $U$ and $V$ be neighbourhoods of 0 and 1 respectively in $M(n)$ and $GL(n)$ such that the exponential map, restricted to $U$, is a homeomorphism from $U$ to $V$. We can, without loss of generality, assume $U$ and $V$ to be symmetric neighbourhoods (neighbourhoods of identity closed under inverse), because we can always replace a general neighbourhood by its intersection with its inverse. We now make an important claim:
Claim. If $G$ is a connected topological group and $V$ is an open neighbourhood of the identity in $G$, then:

$$\bigcup_n V^n = G$$

where $V^n$ denotes the set of $n$-fold products of elements from $V$.

Proof. We first move from $V$ to a symmetric open neighbourhood of the identity contained inside $V$. Let $U = V \cap V^{-1}$. Then, $U$ is a symmetric open neighbourhood of the identity.

Since $U$ is symmetric, the subgroup generated by $U$ is precisely the set

$$\bigcup_n U^n$$

Since this set is a union of open sets, it is open. Hence,

$$\bigcup_n U^n$$

is an open subgroup of the connected topological group $G$.

Let $H$ be this open subgroup. Then, since multiplication by $G$ is a self-homeomorphism of $G$, every coset of $H$ is also an open set. Hence, the union of all the other cosets of $H$ is an open set in $G$, and hence $H$ is both open and closed. This forces $H$ to be either empty or the whole group. But $H$ is a subgroup, so it cannot be empty. Hence $H$ is the whole group. Thus, $\bigcup_n U^n = g$. Since $U \subseteq V$, $\bigcup_n V^n = G$. \qed

Corollary 1. Every element in $G$ can be expressed as a product of exponentials of elements from $g$.

In fact, every element in $G$ can be expressed as a product of exponentials of elements from $U$.

3. Lie group and Lie algebra representations

3.1. Definition of representations. A representation of a Lie group is simply a usual group-theoretic representation with the additional requirement of being continuous and having a closed image.

Definition (Representation of a Lie group). Let $G$ be a Lie group. A representation (defined) of $G$ is a continuous homomorphism from $G$ to $GL_n(k)$ such that the image is a closed subgroup of $GL_n(k)$.

A representation of a Lie algebra is a Lie algebra-theoretic homomorphism to some $M(n)$:

Definition (Representation of a Lie algebra). Let $g$ be a Lie algebra. A representation (defined) of $g$ is a Lie algebra-theoretic homomorphism from $g$ to $M_n(k)$ for some $n$. A Lie algebra-theoretic homomorphism is a linear map of vector spaces that also preserves the Lie bracket operation.

Now, we want to show that representations of a Lie group give rise to representations of its Lie algebra. In other words, we want to show the following:

Claim. If $\pi$ is a Lie group representation of a group $G$ over the vector space $V$, then the differential of $\pi$ at the identity, which we denote by $\dot{\pi}$, gives a Lie algebra representation of the Lie algebra of $G$ (say $g$) over the same vector space $V$.

Further, the $\pi$ composed with the exponential map from $g$ to $G$ equals the exponential map from $M(n)$ to $GL(n)$, composed with $\pi$. 5
The “commutative diagram” postulated in the claim further tells us that any continuous representation is smooth.

In fact, more is true: any measurable homomorphism is smooth. However, we shall not explore this point further.

3.2. A flashback. We had earlier said that there are certain special functions from $SU(2)$ to $\mathbb{C}$ that play a role analogous to the characters of $S^1$, in that the space spanned by them covers a “lot” of functions on $SU(2)$. We can now see what those functions are. The functions are the matrix entries of certain continuous representations of $SU(2)$. More specifically, for each $n$, we define an $(n + 1)$-dimensional representation of $SU(2)$ with a particular basis chosen for the vector space. Thus, each element of $SU(2)$ gives a matrix of order $(n + 1)$. The $(n + 1)^2$ entry functions of this matrix give the functions on $SU(2)$.

The goal (for the next lecture) is to explicitly describe these representations of $SU(2)$, and prove that they are irreducible.
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