PROBLEM SET 7

ANALYSIS II

Problem 1. Suppose $f^{-1}(c)$ is a regular level set of a smooth function $f : A \subset \mathbb{R}^{k+n} \to \mathbb{R}^n$. Take $(a, b) \in f^{-1}(c)$. By the implicit function theorem, there exists a smooth function $g : U \subset \mathbb{R}^k \to \mathbb{R}^n$ on an open set U containing a, such that $g(U) \subset f^{-1}(c)$ and $g(a) = b$. Recall that the tangent space $T_{(a,b)}f^{-1}(c) \subset T_{(a,b)}\mathbb{R}^{k+n}$ is defined to be the orthogonal complement of the row space of $Df(a, b)$ (i.e. the kernel of the linear transformation $y \mapsto Df(a, b)y$). Show that $T_{(a,b)}f^{-1}(c)$ is also equal to the column space of $DG(a)$, where $G : \mathbb{R}^k \to \mathbb{R}^{k+n}$ is defined by $x \mapsto (x, g(x))$.

Problem 2. Give a basis for the tangent space and the normal space of the set S at the point p. Also give equations defining the tangent space and the normal space as a subset of $\mathbb{R}^m$.

(a) Let S be the intersection of the surfaces $x^2+y^2-z^2 = 1$ and $x+y+z = 5$ in $\mathbb{R}^3$. Let $p = (1, 2, 2)$.

(b) Let S be the surface $z = \ln(\sqrt{x^2 + y^2})$, and let $p = (1, -1, \ln(2)/2)$.

Problem 3. Use the method of Lagrange multipliers to

(1) find the minimum value of $x^2 + y^2 + z^2$ subject to the constraints $x + y - z = 0$ and $x + 3y + z = 2$.

(2) find the minimum value of xyz on $f^{-1}(1) \cap \{(x, y, z) : x > 0, y > 0, z > 0\}$, where

$$f(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

Problem 4. (a) Let $f : \mathbb{R}^{2n} \to \mathbb{R}$ be the Euclidean dot product of the first n variables with the last; i.e. $f(x, y) = \langle x, y\rangle$. Use the method of Lagrange multipliers to show that $|f(x, y)| \leq 1$ on the set $S = \{(x, y) : ||x||^2 = ||y||^2 = 1\}$.

(b) Use the previous part to prove the Cauchy Schwartz Inequality. In particular, for arbitrary x and y in $\mathbb{R}^n$, show that $|f(x, y)|^2 \leq ||x||^2 \cdot ||y||^2$.

Problem 5. Show that the function $xyz(x + y + z - 1)$ has one non-degenerate critical point and an infinite set of degenerate critical points. Show that the non-degenerate critical point is a local minimum.

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