PROBLEM SET 2

ANALYSIS II

Problem 1. Let f and g be integrable functions on $[a, b]$.

(a) Show that if P is any partition of $[a, b]$, then

$$U(f + g, P) \leq U(f, P) + U(g, P).$$

Provide a specific example where the inequality is strict. What does the corresponding inequality for lower sums look like?

(b) Prove that $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

Problem 2. Show that products of integrable functions are also integrable as follows:

(a) If f satisfies $|f(x)| \leq M$ on $[a, b]$, show

$$|(f(x))^2 - (f(y))^2| \leq 2M|f(x) - f(y)|.$$

(b) Prove that if f is integrable on $[a, b]$, then so is f^2.

(c) Show that for any integrable functions f and g, the product fg is also integrable. (Hint: consider the square of $(f + g)$).

Problem 3. For each $n \in \mathbb{N}$, let

$$h_n(x) = \begin{cases} 1/2^n & \text{if } 1/2^n < x \leq 1 \\ 0 & \text{if } 0 \leq x \leq 1/2^n, \end{cases}$$

and set $H(x) = \sum_{n=1}^{\infty} h_n(x)$. Show that H is integrable and compute $\int_0^1 H$.

Problem 4 (Integration by parts)**.** (a) Assume $h(x)$ and $k(x)$ have continuous derivatives on $[a, b]$ and derive the familiar integration by parts formula:

$$\int_a^b h(t)k'(t)dt = h(b)k(b) - h(a)k(a) - \int_a^b h'(t)k(t)dt.$$

(b) How can Problem 2 above be used to weaken the hypotheses in part (a).

Problem 5. Given a function f on $[a, b]$, define the *total variation* of f to be

$$Vf = \sup \left\{ \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \right\},$$

where the supremum is taken over all partitions P of $[a, b]$.

(a) If f' exists and is continuous, use the Fundamental Theorem of Calculus to show that $Vf \leq \int_a^b |f'|$.

(b) Use the Mean Value Theorem to establish the reverse inequality and conclude that $Vf = \int_a^b |f'|$.

*All questions taken from *Understanding Analysis: 2nd Edition* by Stephen Abbott.

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