PROBLEM SET 10

ANALYSIS II

Problem 1. Let V be a vector space. We say two ordered bases β and β' for V are *equivalent* if there exists a linear transformation $A : V \to V$ with positive determinant taking β to β'.

(a) Show that there are exactly two equivalence classes of ordered bases for V. We define an *orientation* on V to be a choice of one of these equivalence classes.

(b) Suppose $(V, [\beta])$ is an *oriented vector space*; i.e. a vector space along with a choice of orientation. What are the *orientation preserving* linear transformations from $(V, [\beta])$ to $(V, [\beta])$? What about from $(V, [\beta])$ to $(V, \mathrm{diag}(-1, 1, \ldots, 1)[\beta])$?

(c) More generally, an *orientation* on $\mathbb{R}^n$ is a continuously varying choice of ordered basis for every point in $\mathbb{R}^n$. We say an invertible differentiable map $f : \mathbb{R}^n \to \mathbb{R}^n$ is *orientation preserving* if $Df(p)$ has positive determinant for every p. Give an example of a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ that is not orientation preserving. Is it possible for a map to be orientation preserving in some region of $\mathbb{R}^n$, but not orientation preserving in another region?

Let Σ denote the Riemann sphere, which we realize as the unit sphere in $\mathbb{R}^3$. Identify the xy-plane with the complex plane $\mathbb{C}$ in the standard way. Let N denote the north pole $(0, 0, 1)$. As in class, we define the stereographic projection $\Phi : \mathbb{C} \to \Sigma$ as the map sending a point $p \in \mathbb{C}$ to the point $\hat{p} := (\Sigma \setminus \{N\}) \cap \overline{pN}$, where $\overline{pN}$ denotes the line through p and N.

Problem 2. Prove that Φ takes circles to circles.

Problem 3. Prove that the geometric inversion map $\{z \mapsto \frac{1}{\bar{z}}\}$ corresponds to reflection of the Riemann sphere Σ through the complex plane (i.e the xy-plane). Using the fact that compositions of conformal maps are conformal, show that $\{z \mapsto \frac{1}{z}\}$ is conformal everywhere.

Problem 4. Give an explicit formula for $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$. Give an explicit formula for its inverse.

Problem 5. Let C denote the unit circle in the xy-plane. As mentioned in class, the cylindrical projection of Archimedes is a map from $\Sigma \setminus \{N, S\} \to C \times [-1, 1]$ defined by $(x, y, z) \mapsto (\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z)$. Prove that this map preserves area, assuming the following facts about area of surfaces:

(i) The area of a cylinder is given by the height times the circumference.

(ii) The area of a frustrum of a cone is 2π times the product of the radius and the slant height.

(iii) The area of a small slice of the sphere is approximated by the area of the frustrum of a cone with appropriate radius and slant height.

(You can find a sketch of the proof here.)

Date: April 12, 2018.