DIFFERENTIABLE LIMIT THEOREM

Theorem 0.1. (Differentiable limit theorem) *Suppose* (f_n) *is a sequence of differentiable functions on the interval* $[a, b]$. *Suppose* (f_n) *converges pointwise to a limit* f, *and the sequence of derivatives* (f_n') *converges uniformly to* $g : [a, b] \to \mathbb{R}$. *Then,*

(a) f *is differentiable,*
(b) *and* $f' = g$.

Proof. Consider an arbitrary point c in the interval. The differentiability of f_n at c implies that the function

$$h_n(x) := \begin{cases} \frac{f_n(x)-f_n(c)}{x-c}, & x \neq c, \\ f_n'(c), & x = c. \end{cases}$$

is continuous.

We would like to show that the sequence (h_n) is uniformly Cauchy. We are required to bound

$$|h_m(x) - h_n(x)| = \left| \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right|$$

uniformly for all x for large m, n. The rhs can be expressed in terms of the derivative functions f_m', f_n' by applying the mean value theorem on $f_m - f_n$. Indeed, since $f_m - f_n$ is differentiable, there is a point $\alpha \in [c, x]$ such that

$$\left| \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| = \left| \frac{(f_m(x) - f_n(x)) - (f_m(c) - f_n(c))}{x - c} \right| = |f_m'(\alpha) - f_n'(\alpha)|.$$

The expression $|f_m'(\alpha) - f_n'(\alpha)|$ can be bounded using the fact that (f_n') converges uniformly. A uniformly convergent sequence is also uniformly Cauchy (prove!), and therefore $|f_m'(\alpha) - f_n'(\alpha)|$ is arbitrarily for small if m and n large enough.

The precise details are as follows. Consider an arbitrary $\epsilon > 0$. Since (f_n') is uniformly Cauchy, there exists N such that

$$m, n \geq N \implies |f_m'(x) - f_n'(x)| < \epsilon, \quad \forall x \in [a, b]. \tag{1}$$

From our discussion above, for any $x \neq c$, there is a point α such that

$$|h_m(x) - h_n(x)| = |f_m'(\alpha) - f_n'(\alpha)|,$$

and therefore,

$$m, n \geq N \implies |h_m(x) - h_n(x)| < \epsilon, \quad \forall x \in [a, b] \backslash \{c\}.$$

This bound only applies for $x \neq c$, but it is easy to deal with a single point! We can use the bound (1) to obtain

$$m, n \geq N \implies |h_m(c) - h_n(c)| = |f_m'(c) - f_n'(c)| < \epsilon.$$

We have shown that (h_n) is uniformly Cauchy. Therefore, it uniformly converges to a continuous limit $h : [a, b] \to \mathbb{R}$. Further,

$$h(x) = \begin{cases} \frac{f(x)-f(c)}{x-c}, & x \neq c, \\ \lim_n f_n'(c), & x = c. \end{cases}$$

The first statement follows from the fact that $f_n(x) \to f(x)$, and the algebraic limit theorem. The continuity of h implies that f is differentiable at c, and the derivative is g, which is equal to $\lim_n f_n'(c)$. Since the choice of c is arbitrary, we can conclude that f' is differentiable on $[a, b]$ and $f' = g$.

□