PROBLEM SET 7

INTRO TO REAL ANALYSIS

Problem 1. Recall that Thomae's function $t(x)$ is defined by

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbb{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(a) Construct three different sequences (x_n), (y_n), and (z_n), each of which converges to 1 without using the number 1 as a term in the sequence.
(b) Compute $\lim t(x_n), \lim t(y_n)$, and $\lim t(z_n)$.
(c) Make a conjecture for the value of $\lim_{x \to 1} t(x)$, and prove it using the $\epsilon - \delta$ definition of functional convergence.

Problem 2. We write $\lim_{x \to c} f(x) = \infty$ if for all $M > 0$ we can find a $\delta > 0$ such that whenever $0 < |x - c| < \delta$, it follows that $f(x) > M$.
(a) Show that $\lim_{x \to 0} 1/x^2 = \infty$.
(b) Construct a definition for the statement $\lim_{x \to \infty} f(x) = L$. Show that $\lim_{x \to \infty} 1/x = 0$.

Problem 3. Prove the Squeeze Theorem. That is, let f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. If $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$ at some limit point c of A, show that $\lim_{x \to c} g(x) = L$ as well.

Problem 4. Prove the Contraction Mapping Theorem. That is, let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in \mathbb{R}$.

(1) Show that f is continuous on $\mathbb{R}$.
(2) Pick some point $y_1 \in \mathbb{R}$ and construct a sequence

$$(y_1, f(y_1), f(f(y_1)), \ldots).$$

Writing $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence we can let $y = \lim y_n$.
(3) Prove that y is a fixed point of f and that it is unique in this regard.
(4) Finally, prove that if x is *any* arbitrary point in $\mathbb{R}$, then the sequence $(x, f(x), f(f(x)), \ldots)$ converges to y defined in (b).

The following problem is optional. It will not contribute to or detract from your grade, but you are encouraged to attempt it.

Challenge 1. Let F and U be closed and open sets in $\mathbb{R}$ respectively. Construct functions f and g from $\mathbb{R}$ to $\mathbb{R}$ whose sets of discontinuities are precisely F and U respectively. For hints, see problem 4.3.14 in the textbook.

*All questions taken from *Understanding Analysis: 2nd Edition* by Stephen Abbott.

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