PROBLEM SET 6

INTRO TO REAL ANALYSIS

Problem 1. Given $A \subset \mathbb{R}$, let L be the set of all limit points of A.

(a) Show that the set L is closed.

(b) Argue that if x is a limit point of $A \cup L$, then x is a limit point of A.

Problem 2. Prove that the only sets that are both open and closed are $\mathbb{R}$ and the empty set $\varnothing$.

Problem 3. Let C be the Cantor set. We will prove that the set $C + C = \{x + y : x, y \in C\}$ is equal to the closed interval $[0, 2]$. Clearly $C + C \subset [0, 2]$, so we only need to prove the reverse inclusion. Thus, given $s \in [0, 2]$, we must find two elements $x, y \in C$ satisfying $x + y = s$.

(a) Show that there exist $x_1, y_1 \in C_1$ for which $x_1 + y_1 = s$. Show in general that, for an arbitrary $n \in \mathbb{N}$, we can always find $x_n, y_n \in C_n$ for which $x_n + y_n = s$.

(b) Although (x_n) and (y_n) may not converge, show that they can still be used to produce the desired x and y in C satisfying $x + y = s$.

Problem 4. Let K and L be nonempty compact sets, and define the *distance* between K and L to be

$$d = \inf\{|x - y| : x \in K \text{ and } y \in L\}.$$

(a) If K and L are disjoint, show $d > 0$ and that $d = |x_0 - y_0|$ for some $x_0 \in K$ and $y_0 \in L$.

(b) Show that it's possible to have $d = 0$ if we only assume that the disjoint sets K and L are closed.

Problem 5. Finish the proof of the Heine-Borel Theorem from class. In particular, assume a set K is closed and bounded (and hence that every sequence in K has a subsequence that converges to a limit that is also in K). Prove that any open cover for K has a finite subcover. (Two possible approaches are outlined in problems 3.3.9 and 3.3.10 of Abbott's book.)

The following problem is optional. It will not contribute to or detract from your grade, but you are encouraged to attempt it.

Challenge 1. Repeat the Cantor set construction we did in class, starting with the interval $[0, 1]$. This time however, remove the open middle fourth from each component.

(a) Is the resulting set compact? Perfect?

(b) Using the methods of Section 3.1 of Abbott's book, compute the length and dimension of this Cantor-like set.

*All questions taken from *Understanding Analysis: 2nd Edition* by Stephen Abbott.

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