PROBLEM SET 4

INTRO TO REAL ANALYSIS

Problem 1. (a) Use the Monotone Convergence Theorem to prove the Archimedean Property without making any use of the Axiom of Completeness (i.e. without using the Least Upper Bound Property).

(b) Use the Monotone Convergence Theorem to prove the Nested Interval Property, again without using the AoC.

Problem 2. For each series, find an explicit formula for the sequence of partial sums and determine if the series converges.

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(c) $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$

Problem 3. Let (b_n) be a decreasing sequence of nonnegative terms. Show that if the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then so does the series $\sum_{n=1}^{\infty} b_n$.

Problem 4. (a) Prove that if an infinite series converges, then the associative property holds. In particular, assume $\sum_{n=1}^{\infty} a_n = L$. Show that any regrouping of terms

$$(a_1 + \cdots + a_{n_1}) + (a_{n_1+1} + \cdots + a_{n_2}) + (a_{n_2+1} + \cdots + a_{n_3}) + \cdots$$

leads to a series that also converges to L.

(b) We have seen that addition fails to be associate for the series $\sum_{n=1}^{\infty}(-1)^n$. Explain where the proof in (a) breaks down in this example.

Problem 5. Provide a proof of the Axiom of Completeness (i.e. the Least Upper Bound Property) using only the Nested Interval Property and the property that $(\frac{1}{2^n}) \to 0$. The proof strategy we used for the Bolzano-Weierstrass Theorem may come in handy.

The following problem is optional. It will not contribute to or detract from your grade, but you are encouraged to attempt it.

Challenge 1. Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

(b) The *limit superior* of (a_n), or $\limsup a_n$, is defined by

$$\limsup a_n = \lim y_n,$$

where y_n is the sequence from part (a). Provide a reasonable definition for $\liminf a_n$ and briefly explain why it always exists for any bounded sequence.

(c) Prove that $\liminf a_n \leq \limsup a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

(d) Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. In this case all three share the same value.

*All questions taken from *Understanding Analysis: 2nd Edition* by Stephen Abbott.

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