PROBLEM SET 3

INTRO TO REAL ANALYSIS

Problem 1. (a) Show that $(a, b) \sim \mathbb{R}$ for any interval (a, b).
(b) Show that $(a, \infty) \sim \mathbb{R}$ for any unbounded interval (a, ∞).
(c) Show that $[0, 1) \sim (0, 1)$ by exhibiting a $1-1$ onto function between the sets.

Problem 2. (a) Give an example of a countable collection of disjoint open intervals.
(b) Give an example of an uncountable collection of disjoint open intervals, or argue that no such collection exists.

Problem 3. A real number $x \in \mathbb{R}$ is called *algebraic* if it is a root of a polynomial with integer coefficients; in other words, if there exists integers $a_0, a_1, a_2 \ldots, a_n \in \mathbb{Z}$, not all zero, such that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

Real numbers that are not algebraic are called *transcendental*.
(a) Show that $\sqrt{2}$ and $\sqrt{2} + \sqrt{3}$ are algebraic.
(b) Fix $n \in \mathbb{N}$, and let A_n be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Show that A_n is countable.
(c) Conclude that the set of algebraic numbers is countable (feel free to use a result from class here). What does this imply about the set of transcendental numbers?

Problem 4. (1) Let $C \subset [0, 1]$ be uncountable. Show that there exists $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable.
(2) Now let A be the set of all $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable, and set $\alpha = \text{sup} A$. Is $C \cap [\alpha, 1]$ an uncountable set?
(3) Does the statement in (a) remain true if 'uncountable' is replaced by 'infinite'?

Problem 5. Prove that the limit of a sequence, when it exists, must be unique. To get started, assume that $(a_n) \to a$ and also that $(a_n) \to b$. Now argue $a = b$.

The following problems are optional. They will not contribute to or detract from your grade, but you are encouraged to think about them.

Challenge 1. Construct a $1-1$ function from $\mathbb{R}$ to $P(\mathbb{N})$, the power set of the natural numbers. Construct a $1-1$ function in the reverse direction as well. (By the Schroeder-Bertstein Theorem (see Exercise 1.5.11 of the textbook), this implies that $\mathbb{R} \sim P(\mathbb{N})$).

Challenge 2. Given a set B, a subset $\mathcal{A}$ of the power set $P(B)$ is called an *antichain* if no element of $\mathcal{A}$ is a subset of any other element of $\mathcal{A}$. Does $P(\mathbb{N})$ contain an uncountable antichain?

*All questions taken from *Understanding Analysis: 2nd Edition* by Stephen Abbott.

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