PROBLEM SET 2

INTRO TO REAL ANALYSIS

Problem 1. Let $A \subset \mathbb{R}$ be nonempty and bounded below. Prove that A has a greatest lower bound. (Hint: Define $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$ and show that $\sup B = \inf A$.)

Problem 2. Let $A_1, A_2, A_3, \ldots$ be a collection of nonempty sets, each of which is bounded above.
(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup(\bigcup_{k=1}^{n} A_k)$.
(b) Consider $\sup(\bigcup_{k=1}^{\infty} A_k)$. Does the formula in (a) extend to the infinite case?

Problem 3. Compute (without proofs) the suprema and infima (if they exist) of the following sets:
(a) $\{\frac{m}{n} : m, n \in \mathbb{N} \text{ with } m < n\}$.
(b) $\{\frac{(-1)^m}{n} : m, n \in \mathbb{N}\}$.
(c) $\{\frac{n}{3n+1} : n \in \mathbb{N}\}$.
(d) $\{\frac{m}{m+n} : m, n \in \mathbb{N}\}$.

Problem 4. (a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.
(b) Give an example to show that this is not the case if we only assume $\sup A \leq \sup B$.

Problem 5. Let $A \subset \mathbb{R}$ be nonempty and bounded above, and let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}, s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show that $s = \sup A$.

Problem 6. Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show that $\sup T = b$.

The following problem is optional. It will not contribute to or detract from your grade, but you are encouraged to attempt it.

Challenge 1. The *Cut Property* of the real numbers is the following: If A and B are nonempty, disjoint sets with $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in A$ and $b \in B$, then there exists $c \in \mathbb{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.
(a) Prove that the Cut Property is equivalent to the Axiom of Completeness (i.e. to the statement that any set bounded above has a least upper bound).
(b) Give an example showing that the Cut Property does not hold when $\mathbb{R}$ is replaced by $\mathbb{Q}$.

*All questions taken from *Understanding Analysis: 2nd Edition* by Stephen Abbott.

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