PROBLEM SET 12

INTRO TO REAL ANALYSIS

Problem 1. Let

$$g(x) = \frac{nx + x^2}{2n},$$

and set $g(x) = \lim g_n(x)$. Show that g is differentiable in two ways:

(a) Compute $g(x)$ by algebraically taking the limit as $n \to \infty$ and then find $g'(x)$.

(b) Compute $g_n'(x)$ for each $n \in \mathbb{N}$ and show that the sequence of derivatives (g_n') converges uniformly on every interval $[-M, M]$. Cite the appropriate theorem to conclude that $g'(x) = \lim g_n'(x)$.

Problem 2. Decide whether each proposition is true or false, providing a short justification or counterexample as appropriate.

(a) If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then (g_n) converges uniformly to zero.

(b) If $0 \leq f_n(x) \leq g_n(x)$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

(c) If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, then there exist constants M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$ and $\sum_{n=1}^{\infty} M_n$ converges.

Problem 3. (a) Prove that

$$h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \cdots$$

is continuous on $[-1, 1]$.

(b) Note that the series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

converges for every x in the interval $[-1, 1)$ but does not converge when $x = 1$. For a fixed $x_0 \in (-1, 1)$, explain how we can still use the Weierstrass M-Test to prove that f is continuous at x_0.

Problem 4. (A stronger version of the differentiable limit theorem) Suppose (f_n) is a sequence of differentiable functions defined on an interval $[a, b]$. Suppose the sequence f_n' uniformly convereges to a limit g. Further, assume that there is a point $x_0 \in [a, b]$ for which the sequence $(f_n(x_0))$ is converging. Prove that

(1) the sequence (f_n) uniformly converges to a limit $f : [a, b] \to \mathbb{R}$.

(2) The limit f is differentiable,

(3) and $f' = g$.

HINT: Do not reprove the differentiable limit theorem, try to apply it instead. To prove uniform converegnce of (f_n), try proving that it is uniformly Cauchy (since you do not have a hold on the limit function). The mean value theorem is helpful, and you may route your proof via the estimate

$$\exists \alpha \in [x_0, x] : \quad |f_m(x) - f_n(x)| \leq |f_m(x_0) - f_n(x_0)| + |f_m'(\alpha) - f_n'(\alpha)| \cdot |x - x_0|.$$

Problem 5. Let $\{r_1, r_2, \ldots\}$ be an enumeration of the set of rational numbers. For each $r_n \in \mathbb{Q}$, define

$$u_n(x) = \begin{cases} 1/2^n & \text{for } x > r_n \\ 0 & \text{for } x \leq r_n. \end{cases}$$

Let $h(x) = \sum_{n=1}^{\infty} u_n(x)$. Prove that h is a monotone function defined on all of $\mathbb{R}$ that is continuous at every irrational point.

*All questions taken from *Understanding Analysis: 2nd Edition* by Stephen Abbott.

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