# PROBLEM SET 11

INTRO TO REAL ANALYSIS

**Problem 1.** Let $f$ be uniformly continuous on all of $\mathbb{R}$, and define a sequence of functions by $f_n(x) = f(x + \frac{1}{n})$. Show that $f_n \to f$ uniformly. Give an example to show that this proposition fails if $f$ is only assumed to be continuous and not uniformly continuous on $\mathbb{R}$.

**Problem 2.** Assume $(f_n)$ and $(g_n)$ are uniformly convergent sequences of functions.

(1) Show that $(f_n + g_n)$ is a uniformly convergent sequence of functions.
(2) Give an example to show that the product $(f_n g_n)$ may not converge uniformly.
(3) Prove that if there exists an $M > 0$ such that $|f_n| \le M$ and $|g_n| \le M$ for all $n \in \mathbb{N}$, then $(f_n g_n)$ does converge uniformly.

**Problem 3.** (Dini's theorem) Assume $f_n \to f$ pointwise on a compact set $K$ and assume that for each $x \in K$ the sequence $f_n(x)$ is increasing. Follow these steps to show that if $f_n$ and $f$ are continuous on $K$, then the convergence is uniform.

(a) Set $g_n = f - f_n$ and translate the preceding hypothesis into statements about the sequence $(g_n)$.
(b) Let $\epsilon > 0$ be arbitrary, and define $K_n = \{x \in K : g_n(x) \ge \epsilon\}$. Argue that $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$, and use this observation to finish the argument.

**Problem 4.** Assume $f_n \to f$ pointwise on $[a, b]$ and the limit function $f$ is continuous on $[a, b]$. If each $f_n$ is increasing (but not necessarily continuous), show $f_n \to f$ uniformly.
Hint: We need to find an $N$ such that

$$n > N \implies |f_n(x) - f(x)| < \epsilon \quad \forall x.$$

To find $N$, divide the interval $[a, b]$ into sub-intervals $a = t_0 < t_1 < \cdots < t_n = b$, such that for any $i$, $|f(y_1) - f(y_2)| < \epsilon/3$ for all $y_1, y_2 \in [t_i, t_{i+1}]$.

**Problem 5.**

(1) Define $f_0(x) = x$ for all $x \in [0, 1]$. Now, let

$$f_1(x) = \begin{cases} 3x/2, & \text{for } 0 \le x \le 1/3, \\ 1/2, & \text{for } 1/3 < x < 2/3, \\ 3x/2 - 1/2, & \text{for } 2/3 \le x \le 1. \end{cases}$$

Sketch $f_0$ and $f_1$ over $[0, 1]$ and observe that $f_1$ is continuous, increasing, and constant on the middle third $(1/3, 2/3) = [0, 1]\backslash C_1$.

(2) Construct $f_2$ by imitating this process of flattening out the middle third of each nonconstant segment of $f_1$. Specifically, let

$$f_2(x) = \begin{cases} \frac{1}{2} f_1(3x), & \text{for } 0 \le x \le 1/3, \\ f_1(x), & \text{for } 1/3 < x < 2/3, \\ \frac{1}{2} f_1(3x - 2) + \frac{1}{2}, & \text{for } 2/3 \le x \le 1. \end{cases}$$

If we continue this process, show that the resulting sequence $(f_n)$ converges uniformly on $[0, 1]$.

(3) Let $f = \lim f_n$. Prove that $f$ is a continuous, increasing function on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$ that satisfies $f'(x) = 0$ for all $x$ in the open set $[0, 1]\backslash C$. Recall that the "length" of the Cantor set $C$ is 0. Somehow, $f$ manages to increase from 0 to 1 while remaining constant on a set of "length 1."

**The following problem is optional. It will not contribute to or detract from your grade, but you are encouraged to attempt it.**

*Date*: November 6, 2017.

**Challenge 1.** (Arzela-Ascoli theorem) Suppose $(f_n)$ is a sequence of functions on $[0,1]$ satisfying the following conditions:

(1) the sequence is *uniformly bounded*, i.e. there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in [0,1]$, and for all $n \in \mathbb{N}$.
(2) The sequence is *equicontinuous*, i.e. for any $\epsilon > 0$, there is a $\delta > 0$ such that for all $x, y \in [0,1]$ and $n \in \mathbb{N}$,

$$|x - y| < \delta \implies |f_n(x) - f_n(y)| < \epsilon.$$

Then, the sequence $(f_n)$ has a uniformly converging subsequence. Consult Problem 6.2.15 in the book for hints.

*All questions taken from *Understanding Analysis: 2nd Edition* by Stephen Abbott.