PROBLEM SET 9, VERSION 2

INTRODUCTION TO MANIFOLDS

Some typos were noticed in the previous version. Corrections have been marked in red.

Problem 1. In order to get practice with pullbacks of forms, do problems 19.1 - 19.4 in [Tu]. Do not submit these.

Problem 2. (1) Let $x^1, \dots, x^n, y^1, \dots, y^n$ be a basis for $(\mathbb{R}^{2n})^*$. Prove that the 2-covector $\sum_{j=1}^n dx^j \wedge dy^j$ is nondegenerate.

(2) Let M be a smooth manifold. Prove that $d\lambda$ is nondegenerate, where λ is the Liouville 1-form on T^*M. (The 2-form $-d\lambda$ is known as the *standard symplectic form* on T^*M. Since a *symplectic* form is a closed, nondegenerate 2-form, this terminology is justified.)

Problem 3. (1) Suppose 0 is a regular value of $f(x, y) \in C^\infty(\mathbb{R}^2)$. Construct a nowhere-vanishing 1-form on the one-dimensional submanifold $f^{-1}(0)$.

(2) Suppose 0 is a regular value of $f(x, y, z) \in C^\infty(\mathbb{R}^3)$. Construct a nowhere-vanishing 2-form on the two-dimensional submanifold $f^{-1}(0)$. Hint: first show that the equalities

$$\frac{dx \wedge dy}{f_z} = \frac{dy \wedge dz}{f_x} = \frac{dz \wedge dx}{f_y}$$

hold on $f^{-1}(0)$ whenever they make sense.

Problem 4. Let ω be a differential form, X a vector field, and f a smooth function on a manifold M. Recall that the Lie derivative $\mathfrak{L}_X\omega$ is not $C^\infty(M)$-linear in either variable. However, using Cartan's homotopy formula $\mathfrak{L}_X = d\iota_X + \iota_X\omega$, prove that the Lie derivative does satisfy:

$$\mathfrak{L}_{fX}\omega = f\mathfrak{L}_X\omega + df \wedge \iota_X\omega.$$

Problem 5. Prove that $[\mathfrak{L}_X, \iota_Y] = \iota_{[X,Y]}$ for any $X, Y \in \mathfrak{X}(M)$. (For hints see Problem 20.8 in [Tu].)

Problem 6. Let $\omega = dx^1 \wedge \ldots \wedge dx^n \in \Omega^n(\mathbb{R}^n)$ and $X = \sum x^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(\mathbb{R}^n)$ be the volume form and radial vector field respectively. Compute the contraction $\iota_X\omega$.

Problem 7. Consider the unit 2-sphere $S^2 \subset \mathbb{R}^3$. Let $\omega = xdy \wedge dz - ydx \wedge dy + zdx \wedge dy \in \Omega^2(S^2)$ and $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \in \mathfrak{X}(S^2)$. Compute the Lie derivative $\mathfrak{L}_X\omega$.

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