PROBLEM SET 8

INTRODUCTION TO MANIFOLDS

Problem 1. (a) Let S be a regular submanifold of codimension k in a smooth manifold M of dimension n. Let $(U, \phi) = (U, f^1, \ldots, f^n)$ be local coordinates about a point $p \in S$ such that $S \cap U$ is defined by the vanishing of $f^1, \ldots, f^k$. Show that $T_pS = \bigcap_{i=1}^k \ker(df^i)$.

(b) Note that a vector field X on M restricts to a vector field on a submanifold $S \subset M$ if $X_p \in T_pS$ for any $p \in S$. Construct a vector field on $\mathbb{R}^{2n}$ that restricts to a nowhere-vanishing vector field on the unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$.

Problem 2. Let M be a smooth manifold, and let $\pi : T^*M \to M$ is the standard projection from the cotangent bundle over M to M. Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart on M. Then there exist $c_1, \ldots, c_n \in C^\infty(\pi^{-1}U)$ such that for any $\alpha \in \pi^{-1}U$, we have

$$\alpha = \sum_{i=1}^{n} c_i(\alpha) dx^i|_{\pi(\alpha)}.$$

Note that the induced charts

$$(\pi^{-1}U, \tilde{\phi}) := (\pi^{-1}U, x^1 \circ \pi, \ldots, x^n \circ \pi, c_1, \ldots, c_n)$$

give a smooth manifold structure to T^*M for which the projection π is smooth (see e.g. [Tu] or [Lee] for details).

Recall that the Liouville form $\lambda \in \Omega^1(T^*M)$ is defined by $\lambda_{\omega_p} X_{\omega_p} := \omega_p(\pi_* X_{\omega_p})$. Find a formula for λ on $\pi^{-1}U$ in terms of the local coordinates, and use it to show that λ is smooth.

Problem 3. Let G be a Lie group of dimension n with Lie algebra $\mathfrak{g}$. For each $g \in G$, let $c_g := l_g \circ r_{g^{-1}} : G \to G$ be the corresponding conjugation map. Note that the differential at the identity $c_{g*} : \mathfrak{g} \to \mathfrak{g}$ is a linear isomorphism, and hence $c_{g*} \in GL(\mathfrak{g})$. Show that $Ad : G \to GL(\mathfrak{g})$ defined by $Ad(g) = c_{g*}$ is a homomorphism of Lie groups; i.e. that it is an abstract group homomorphism *and* a smooth map between manifolds.

Problem 4. Let G be a compact, connected Lie group of dimension n with Lie algebra $\mathfrak{g}$. Prove that every left-invariant n-form on G is right-invariant. (For hints see problem 18.9 in [Tu].)

Problem 5. Using the fact that the pullback of a smooth k-form is a smooth k-form, prove that if $\pi : \tilde{M} \to M$ is a surjective submersion, then the pullback map $\pi^* : \Omega^*(M) \to \Omega^*(\tilde{M})$ is an injective algebra homomorphism.

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