PROBLEM SET 7

INTRODUCTION TO MANIFOLDS

Problem 1. Prove that an open subgroup H of a connected Lie group G is equal to G.

Problem 2. Show that the differential of the determinant map at $A \in GL(n, \mathbb{R})$ is given by

$$\det_{*,A}(AX) = \det(A)\operatorname{trace}(X)$$

for $X \in \mathbb{R}^{n\times n}$

Problem 3. Prove that

(a) the real orthogonal group $O(n, \mathbb{R})$ is compact, but
(b) the complex orthogonal group $O(n, \mathbb{C})$ is not compact.

Problem 4. Show that the special unitary group $SU(2) \subset GL(2, \mathbb{C})$ is diffeomorphic to the three-sphere. (There are several approaches to this problem. For example, one approach is outlined in problem 15.13 of [Tu]. We will discuss at least two other approaches in class later.)

Problem 5. Let $A \in \mathfrak{gl}(n, \mathbb{R})$ and let $\tilde{A}$ be the left-invariant vector field on $GL(n, \mathbb{R})$ generated by A. Show that $c(t) := e^{tA}$ is the integral curve of $\tilde{A}$ starting at the identity matrix I. Find the integral curve of $\tilde{A}$ starting at $g \in GL(n, \mathbb{R})$.

The following problem is optional. It will not contribute to or detract from your grade.

Challenge 1. Let G be a Lie group. Construct a subgroup $H \subset G$ that is an immersed submanifold of G but *not* a Lie subgroup.

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