PROBLEM SET 7

INTRODUCTION TO MANIFOLDS

Problem 1. Compute $\exp \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Problem 2. Prove that
(a) the real orthogonal group $O(n, \mathbb{R})$ is compact, but
(b) the complex orthogonal group $O(n, \mathbb{C})$ is not compact.

Problem 3. Show that the special unitary group $SU(2) \subset GL(2, \mathbb{C})$ is diffeomorphic to the three-sphere. (There are several approaches to this problem. For example, one approach is outlined in problem 15.13 of [Tu]. Another approach is to consider the action of $SU(2)$ on $S^3 \subset \mathbb{C}^2$ and apply the *Homogeneous Space Theorem* described in class (see [Thm 7.19, Lee] for proof). Yet another approach is to check that quaternions $\mathbb{H}$ can be thought of as a complex vector space with basis $\{1 + i, j + k\}$, and then show that right multiplication by a unit quaternion is a unitary map.)

Problem 4. Let $\mathbb{H}$ be the skew-field of quaternions. Show that the symplectic group $Sp(n, \mathbb{H}) = \{A \in GL(n, \mathbb{H}) : \bar{A}^T A = I\}$ is a regular submanifold of $GL(n, \mathbb{H})$ and compute its dimension.

Problem 5. Let $A \in \mathfrak{gl}(n, \mathbb{R})$ and let $\tilde{A}$ be the left-invariant vector field on $GL(n, \mathbb{R})$ generated by A. Show that $c(t) := e^{tA}$ is the integral curve of $\tilde{A}$ starting at the identity matrix I. Find the integral curve of $\tilde{A}$ starting at $g \in GL(n, \mathbb{R})$.

Problem 6. A manifold whose tangent bundle is trivial is said to be *parallelizable*. Prove that any Lie group G is parallelizable.

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