PROBLEM SET 6

INTRODUCTION TO MANIFOLDS

Problem 1. (a) Let X be the vector field $x\frac{d}{dx}$ on $\mathbb{R}$. For each $p \in \mathbb{R}$, find the maximal integral curve $c(t)$ of X with $c(0) = p$.

(b) Let X be the vector field $x^2\frac{d}{dx}$ on $\mathbb{R}$. For each $p \in \mathbb{R}_{>0}$, find the maximal integral curve $c(t)$ of X with $c(0) = p$.

Problem 2. Consider two smooth vector fields X, Y on $\mathbb{R}^n$:

$$X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}, \qquad Y = \sum_{i=1}^{n} b^i \frac{\partial}{\partial x^i},$$

where a^i, b^i are smooth functions on $\mathbb{R}^n$. Since $[X, Y]$ is also a smooth vector field on $\mathbb{R}^n$,

$$[X, Y] = \sum_{k=1}^{n} c^k \frac{\partial}{\partial x^k}$$

for some smooth functions c^k on $\mathbb{R}^n$. Find a formula for c^k in terms of a^i and b^i.

Problem 3. Let $F : N \to M$ be a smooth diffeomorphism manifolds. Prove that if X and Y are smooth vector fields on N, then

$$F_*[X, Y] = [F_*X, F_*Y].$$

Problem 4. Compute $\exp \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Problem 5. A manifold whose tangent bundle is trivial is said to be *parallelizable*. Prove that any Lie group G is parallelizable.

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