PROBLEM SET 5

INTRODUCTION TO MANIFOLDS

A smooth map $F : N \to M$ is *transverse* to a submanifold[1] $S \subset M$ if for every $p \in F^{-1}(S)$,

$$F_{*,p}(T_pN) + T_{F(p)}S = T_{F(p)}M.$$

We have the following theorem.

Theorem 1. *Suppose the smooth map $F : N \to M$ is transverse to a submanifold S of codimension k in M. Then $F^{-1}(S)$ is a submanifold of codimension k in N.*

Problem 1. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x, y) = x^3 + xy + y^3 + 1.$$

Find all values $c \in \mathbb{R}$ for which the level set $F^{-1}(c)$ is a submanifold of $\mathbb{R}^2$.

Problem 2. Prove that the solution set of the system of equations

$$x^3 + y^3 + z^3 = 1, z = xy,$$

is a submanifold of $\mathbb{R}^3$.

Problem 3. Suppose that a subset S of $\mathbb{R}^2$ has the property that locally on S one of the coordinates is a C^∞ function of the other coordinate. Show that S is a submanifold of $\mathbb{R}^2$.

Problem 4. Let M be a smooth manifold. Show that a submanifold of M is closed in M if and only if the inclusion map is proper.

Problem 5. Suppose the smooth map $F : N \to M$ is transverse to a submanifold S of codimension k in M. Let $p \in F^{-1}(S)$ and let $(U, x^1, \ldots, x^m)$ be an adapted chart centered at $F(p)$ for M relative to S such that $U \cap S = Z(x^{m-k+1}, \ldots, x^m)$, the zero set of the functions $x^{m-k+1}, \ldots, x^m$. Define $G : U \to \mathbb{R}^k$ by $G = (x^{m-k+1}, \ldots, x^m)$.
(a) Show that $F^{-1}(U) \cap F^{-1}(S) = (G \circ F)^{-1}(0)$.
(b) Show that $F^{-1}(U) \cap F^{-1}(S)$ is a regular level set of the function $G \circ F : F^{-1}(U) \to \mathbb{R}^k$.
(c) Prove Theorem 1.

The following two problems are optional. They will not contribute to or detract from your grade, but you are encouraged to attempt them.

We say submanifolds N_1 and N_2 of a smooth manifold M have *transverse intersection* if $T_pN_1 + T_pN_2 = T_pM$ for any $p \in N_1 \cap N_2$. In this case we write $N_1 \pitchfork N_2$.

Challenge 1. Let V be a vector space, and let Δ be the diagonal of $V \times V$. For a linear map $A : V \to V$, consider the graph $W = \{(v, Av) : v \in V\}$. Show that $W \pitchfork \Delta$ if and only if $+1$ is not an eigenvalue of A.

Challenge 2. Let M be a smooth manifold and let $F : M \to M$ be a smooth map with fixed point $p \in M$. If $+1$ is not an eigenvalue of $F_{*,p} : T_pM \to T_pM$, then p is called a *Lefschetz fixed point* of F. We say F is *Lefschetz* if all its fixed points are Lefschetz. Prove that if M is compact and F is Lefschetz, then F has finitely many fixed points.

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[1]By *submanifold* we always mean *regular submanifold* in Tu's terminology.