PROBLEM SET 4 (VERSION 2)

INTRODUCTION TO MANIFOLDS

Suppose a discrete group G acts on a smooth manifold M. The action is *smooth* if the map $p \mapsto g.p$ is a diffeomorphism of M for any $g \in G$. The action is *free* if whenever $g.p = p$ for some $p \in M$, we have $g = 1$. The action is *proper* if the map $G \times M \to M \times M$ defined by $(g, p) \mapsto (g.p, p)$ is proper. Equivalently, the action is proper if and only if any two points $p, p' \in M$ have open neighborhoods U, U' such that the set $\{g \in G : (g.U) \cap U' \neq \varnothing\}$ is finite (see e.g. [Lemma 7.11, Intro. to Smooth Manifolds, John M. Lee]). We have the following theorem. Update: A correction has been marked in red.

Theorem 1 (Quotient by discrete group action)**.** *Let G be a discrete group that acts smoothly, freely, and properly on a smooth manifold M. Let $\pi : M \to M/G$ be the quotient map sending $p \mapsto G.p$. Then*

(a) The topological quotient M/G is Hausdorff and second countable.
(b) Any point $p \in M$ is covered by a smooth chart (U_p, ϕ_p) for which $g.U_p \cap U_p = \varnothing$ for all $g \in G$. Hence $\mathfrak{A} := \{(\pi(U_p), \phi_p \circ (\pi|_{U_p})^{-1})\}_{p \in M}$ is a well-defined collection of charts on M/G.
(c) The collection $\mathfrak{A}$ is a smooth atlas on M/G.

Problem 1. Prove Theorem 1.

Problem 2. Given smooth manifolds M and N, let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ be the two projections. Prove that for any $(p, q) \in M \times N$,

$$\pi_{1*} \oplus \pi_{2*} : T_{(p,q)}(M \times N) \to T_pM \oplus T_qN$$

is an isomorphism.

Problem 3. Let G be a Lie group with multiplication map $\mu : G \times G \to G$, inverse map $\iota : G \to G$, and identity e.

(a) Show that $\mu_{*,(e,e)}(X, Y) = X + Y$ for any $(X, Y) \in T_eG \oplus T_eG \cong T_{(e,e)}(G \times G)$.
(b) Show that $\iota_{*,e}(X) = -X$ for any $X \in T_eG$.

Problem 4. Let M be a smooth manifold. A function $f : M \to \mathbb{R}$ is said to be a *local maximum* if there is a neighborhood U of p such that $f(p) \geq f(q)$ for all $q \in U$.

(a) Let I be an open interval in $\mathbb{R}$. Prove that if a differentiable function $f : I \to \mathbb{R}$ has a local maximum at $p \in I$, then $f'(p) = 0$.
(b) Using (a), prove that a local maximum of a smooth function $f : M \to \mathbb{R}$ is a critical point of f.

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