PROBLEM SET 3

INTRODUCTION TO MANIFOLDS

Problem 1. (a) Show that the 'line with two origins' is locally Euclidean and second countable, but not Hausdorff. (This pathological space is defined by quotienting $\{(x, y) \in \mathbb{R}^2 : y = \pm 1\}$ by the equivalence relation $(x, -1) \sim (x, +1)$ for all $x \neq 0$.)

(b) Show that the disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second countable.

Problem 2. Let S^n denote the unit sphere in $\mathbb{R}^{n+1}$, with north pole $N := (0, \ldots, 0, 1)$ and south pole $S := (0, \ldots, -1)$. Let $p_N : S^n \setminus \{N\} \to \mathbb{R}^n$ be the *stereographic projection* defined by

$$(x^1, \ldots, x^{n+1}) \mapsto \frac{(x^1, \ldots, x^n)}{1 - x^{n+1}}.$$

Let $p_S : S^n \setminus \{S\} \to \mathbb{R}^n$ be defined by $p_S(\mathbf{x}) = p_N(-\mathbf{x})$.

(a) Show that p_N is bijective, with inverse

$$p_N^{-1}(y_1, \ldots, y_n) = \frac{2y^1, \ldots, 2y^n, |y|^2 - 1}{|y|^2 + 1}.$$

(b) Compute the transition map $p_S \circ p_N^{-1}$. Verify that the atlas

$$\{(S^n \setminus \{N\}, p_N), (S^n \setminus \{S\}, p_S)\}$$

defines a differentiable structure on S^n.

Problem 3. Let $\mathbb{RP}^n$ denote the *set* of lines through the origin in $\mathbb{R}^{n+1}$. Let $[x^1 : \ldots : x^{n+1}]$ denote the equivalence class of vectors in $\mathbb{R}^{n+1}$ related by scalar multiplication (hence $[x^1 : \ldots : x^{n+1}] = [\lambda x^1 : \ldots : \lambda x^{n+1}]$ for any $\lambda \in \mathbb{R}^*$). Let $U_i := \{[x^1 : \ldots : x^{n+1}] : x^i \neq 0\} \subset \mathbb{RP}^n$, and let $\phi_i : U_i \to \mathbb{R}^n$ be defined by $[x^1 : \ldots : x^{n+1}] \mapsto (\frac{x^1}{x^i}, \ldots, \widehat{\frac{x^i}{x^i}}, \ldots, \frac{x^{n+1}}{x^i})$, for $i = 1, \ldots, n+1$. Finally, define a collection of subsets of $\mathbb{RP}^n$,

$$\mathcal{B} := \{\phi_i^{-1}(V) : V \subset \mathbb{R}^n \text{ is open }, 1 \leq i \leq n+1\}.$$

(a) Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{RP}^n$.

(b) Show that $\mathbb{RP}^n$ is second countable and Hausdorff under this topology.

(c) Show that the set of charts $\{(U_i, \phi_i)\}$ forms a C^∞-atlas on $\mathbb{RP}^n$. Hence $\mathbb{RP}^n$ is a smooth manifold.

Problem 4. Consider the following C^∞-atlases on $\mathbb{R}$, each consisting of a single chart:

$$\mathfrak{A} := \{(\mathbb{R}, \phi : x \mapsto x)\}, \text{ and } \mathfrak{B} := \{(\mathbb{R}, \psi : x \mapsto x^{1/3})\}.$$

(a) Show that the resulting differentiable structures on $\mathbb{R}$ are distinct (i.e. that the induced maximal atlases are distinct).

(b) Show that there is a diffeomorphism between the smooth manifolds $(\mathbb{R}, \mathfrak{A})$ and $(\mathbb{R}, \mathfrak{B})$.

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