

PROBLEM SET 3

INTRODUCTION TO MANIFOLDS

- Problem 1.** (a) Show that the ‘line with two origins’ is locally Euclidean and second countable, but not Hausdorff. (This pathological space is defined by quotienting $\{(x, y) \in \mathbb{R}^2 : y = \pm 1\}$ by the equivalence relation $(x, -1) \sim (x, +1)$ for all $x \neq 0$.)
- (b) Show that the disjoint union of uncountably many copies of \mathbb{R} is locally Euclidean and Hausdorff, but not second countable.

Problem 2. Let S^n denote the unit sphere in \mathbb{R}^{n+1} , with north pole $N := (0, \dots, 0, 1)$ and south pole $S := (0, \dots, -1)$. Let $p_N : S^n \setminus \{N\} \rightarrow \mathbb{R}^n$ be the *stereographic projection* defined by

$$(x^1, \dots, x^{n+1}) \mapsto \frac{(x^1, \dots, x^n)}{1 - x^{n+1}}.$$

Let $p_S : S^n \setminus \{S\} \rightarrow \mathbb{R}^n$ be defined by $p_S(\mathbf{x}) = p_N(-\mathbf{x})$.

- (a) Show that p_N is bijective, with inverse

$$p_N^{-1}(y_1, \dots, y_n) = \frac{2y^1, \dots, 2y^n, |y|^2 - 1}{|y|^2 + 1}.$$

- (b) Compute the transition map $p_S \circ p_N^{-1}$. Verify that the atlas

$$\{(S^n \setminus \{N\}, p_N), (S^n \setminus \{S\}, p_S)\}$$

defines a differentiable structure on S^n .

Problem 3. Let $\mathbb{R}\mathbb{P}^n$ denote the *set* of lines through the origin in \mathbb{R}^{n+1} . Let $[x^1 : \dots : x^{n+1}]$ denote the equivalence class of vectors in \mathbb{R}^{n+1} related by scalar multiplication (hence $[x^1 : \dots : x^{n+1}] = [\lambda x^1 : \dots : \lambda x^{n+1}]$ for any $\lambda \in \mathbb{R}^*$). Let $U_i := \{[x^1 : \dots : x^{n+1}] : x^i \neq 0\} \subset \mathbb{R}\mathbb{P}^n$, and let $\phi_i : U_i \rightarrow \mathbb{R}^n$ be defined by $[x^1 : \dots : x^{n+1}] \mapsto (\frac{x^1}{x^i}, \dots, \frac{x^i}{x^i}, \dots, \frac{x^{n+1}}{x^i})$, for $i = 1, \dots, n+1$. Finally, define a collection of subsets of $\mathbb{R}\mathbb{P}^n$,

$$\mathcal{B} := \{\phi_i^{-1}(V) : V \subset \mathbb{R}^n \text{ is open}, 1 \leq i \leq n+1\}.$$

- (a) Show that \mathcal{B} is a basis for a topology on $\mathbb{R}\mathbb{P}^n$.
- (b) Show that $\mathbb{R}\mathbb{P}^n$ is second countable and Hausdorff under this topology.
- (c) Show that the set of charts $\{(U_i, \phi_i)\}$ forms a C^∞ -atlas on $\mathbb{R}\mathbb{P}^n$. Hence $\mathbb{R}\mathbb{P}^n$ is a smooth manifold.

Problem 4. Consider the following C^∞ -atlases on \mathbb{R} , each consisting of a single chart:

$$\mathfrak{A} := \{(\mathbb{R}, \phi : x \mapsto x)\}, \text{ and } \mathfrak{B} := \{(\mathbb{R}, \psi : x \mapsto x^{1/3})\}.$$

- (a) Show that the resulting differentiable structures on \mathbb{R} are distinct (i.e. that the induced maximal atlases are distinct).
- (b) Show that there is a diffeomorphism between the smooth manifolds $(\mathbb{R}, \mathfrak{A})$ and $(\mathbb{R}, \mathfrak{B})$.