PROBLEM SET 10, VERSION 2

INTRODUCTION TO MANIFOLDS

Problem 1. Construct an oriented atlas on S^1.

Problem 2. Let M be a smooth manifold. Prove that the tangent bundle TM is orientable. (Hint: either exhibit an oriented atlas as in Problem 21.9 of [Tu], or construct a nonwhere-vanishing top form.)

Problem 3. Orient the unit sphere S^n in R^{n+1} as the boundary of the closed unit ball. Let $U = \{x \in S^n : x^{n+1} > 0\}$ be the upper hemisphere. Note that $(U, \pi) := (U, x^1, \dots, x^n)$ is a coordinate chart on S^n.

(a) Show that the projection $\pi : U \to \mathbb{R}^n$ is orientation-preserving if and only if n is even.
(b) Show that the antipodal map $a : S^n \to S^n$ is orientation-preserving if and only if n is odd.
(c) Show that $\mathbb{R}P^n$ is orientable if n is odd.

Problem 4. Suppose M is a connected non-orientable smooth manifold. Consider the set

$$\tilde{M} := \bigsqcup_{p \in M} \{\text{orientations of } T_pM\},$$

and let $\pi : \tilde{M} \to M$ be the projection $[\omega_p] \mapsto p$. It turns out that $\tilde{M}$ is a connected orientable smooth manifold and that π is a local diffeomorphism. Describe an oriented smooth atlas on $\tilde{M}$, and prove that the transition maps on a nonempty intersection are indeed orientation-preserving ($\tilde{M}$ is a 2-sheeted covering space known as the *orientation covering* of M.)

Problem 5. Let $T^2 = \{(w, x, y, z) : w^2 + x^2 = y^2 + z^2 = 1\} \subset \mathbb{R}^4$ be the oriented torus with orientation form $(-xdw + wdx) \wedge (-zdy + ydz)$. Let $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ be the oriented sphere with orientation form $xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$. Compute the integrals

(a) $\int_{T^2} wydx \wedge dz$,
(b) $\int_{S^2} xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$, and
(c) $\int_{S^2} xzdy \wedge dz + yzdz \wedge dx + x^2 dx \wedge dy$.

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