Syzygies of Koszul algebras

CMI
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(1) k a field
$S = k[a_1, \dots, a_e]$ deg $a_i = 1$; standard graded poly. ring
$R = S/(f_1, \dots, f_c)$ deg $f_i \geq 2$

$\beta^S(R) = (\beta^S_{ij}(R))$ – Betti table of R/S

$\beta^S_{ij}(R) = \operatorname{rank}_k \operatorname{Tor}^S_i(k, R)_j$

<u>Problem</u>: What can one learn about R from $\beta^S(R)$ & vice-versa?

For instance, from $\beta^S(R)$ one can compute:
- dim R (from Hilbert series)
- depth R (Auslander-Buchsbaum)

One can detect if R is
- regular
- c.i.
- Gorenstein
- C.M.

(2) <u>Koszul algebras</u> Given an R-module M, set

$$t^R_i(M) = \max\{j \mid \operatorname{Tor}^R_i(k, M)_j \neq 0\}$$

<u>Fact</u>: $t^R_i(k) \geq i$ for $i \leq \operatorname{pd}_R k$.

<u>Defn</u>: R is Koszul if equality holds for all i.

Thus, R is Koszul $\iff$ $\beta^R(k)$ looks like:

*	*	*	*	...

0

i.e. the minimal resolution of k/R is linear.

<u>Why care</u>: (i) R Koszul $\Rightarrow$ $P_k^R(t) = H_R(-t)^{-1}$, and hence rational

(Avramov–Eisenbud) $\Downarrow$

$$\operatorname{reg}_R M < \infty \quad \forall \quad R\text{-modules } M$$

(Explain in terms of Betti tables)

<u>Note</u>: $\beta_2^R(k) :=$ records the generators of R i.e. (f_i).

To be precise, $\beta_{2j}^R(k) = \#\{i \mid \deg(f_i) = j\}$

$\therefore$ R Koszul $\Rightarrow$ $\deg(f_j) = 2 \quad \forall \ j$

(Converse is false)

<u>Examples</u>: (1) quadratic monomials define Koszul algebras

(2) Ideals admitting a quadratic Gröbner basis

(3) Many 'naturally' occurring rings are Koszul.

(See, for instance, 'Koszul algebras & regularity', by Conca, De Negri, Rossi)

Notoriously difficult to detect Koszul property:

- $\frac{k[x,y,u,v]}{(x^2,y^2,u^2,v^2,xy+uv)}$ not Koszul $\beta^R_{34}(k) \neq 0$

Roos has constructed an example of [illegible].

$$R = \frac{k[x,y,z,u,v,w]}{(x^2, xy, yz, z^2, zu, u^2, uv, vw, w^2, xz+\alpha zw-uw, zw+xu+(\alpha-2)uw)}$$

char $k=0$, $\alpha \geq 2$ integer.

Then $t^R_{ij}(k) = 0$ for $j \neq i$, $i \leq \alpha$

$t^R_{\alpha+1,\alpha+2}(k) \neq 0$.

- They all have the same Betti table.

<u>Question</u>: How does the Koszul property impact $\beta^S(R)$?

<u>Theorem</u>: (Avramov, Conca, Iyengar, MRL (2010), preprint 2012)

Suppose R is Koszul. Then

(i) $t^S_i(R) \leq 2i$ $\forall i$ (Backelin -80 unpublished)

(ii) Inequality is strict for $i >$ codim R = dim S - dim R

- In particular, $\mathrm{reg}_S R \leq \mathrm{pd}_S R$

Equality holds $\iff$ R is ci (of quadrics)

<u>Discussion</u>: - One cannot reconstruct $\beta^S(R)$ from $\beta^R(k)$.

There is a relationship:

$$\mathrm{Tor}^R_p(k,k) \otimes_k \mathrm{Tor}^S_q(k,R) \Rightarrow \mathrm{Tor}^S_{p+q}(k,k)$$

[There are delta spectral sequences].

There is however a close relationship betw algebra resolution of R/S and that of k/R.

(III) DG (= differential graded) algebra resolutions:

- R any commutative ring

Given a graded set $X = \{X_i\}_{i\geq 1}$ set

$$R[X] := \mathrm{Sym}_R(X_{even}) \otimes_R \Lambda_R(X_{odd})$$

A free dg R-algebra $:= (R[X], \partial)$ - $\partial^2 = 0$

- $\partial(ab) = \partial(a)b + (-1)^{|a|} a\partial b$

Defn: Given $R \twoheadrightarrow R/I$, a <u>dg model</u> of R/I is

a free dg R-algebra $(R[X], \partial)$ equipped with a morphism $R[X] \xrightarrow[\simeq]{} R/I$.

Examples ① Say $(g_1, \ldots, g_d) \subseteq R$ is a regular sequence.

Then $R \twoheadrightarrow R/(\underline{g})$ has a DG model:

$$(R[x_1, \ldots, x_d], \partial) \quad \cancel{\deg x_i} \; |x_i| = 1 \;\; \forall i$$

$$\partial(x_i) = g_i$$

i.e. the Koszul cx. on $\underline{g}$.

② Say $(\underline{f}) \subseteq (\underline{g}) \subseteq S$ regular sequences
$(f_1, \ldots, f_c) \subseteq (g_1, \ldots, g_d)$

$$R = S/(\underline{f}) \twoheadrightarrow S/(\underline{g})$$

Write: $f_j = \sum_i \lambda_{ij} g_i \quad \lambda_{ij} \in S$.

~~Then~~ Tate: $R[x_1, \ldots, x_d, y_1, \ldots, y_c]$ $|x_i| = 1$, $|y_j| = 2$

$$\partial(x_i) = g_i$$
$$\partial(y_j) = \sum \lambda_{ij} x_i$$

is a DG model for $S/(\underline{g})$ over R.
[Char $k = 0$!!]

Now back to our context:

$S = k[\underline{a}]$ $\quad R = k[\underline{a}]/(\underline{f})$ std. graded k-algebra.

A free dg R-algebra $(R\langle X\rangle, \partial)$ is minimal if

$$\partial(X) \subseteq (\underline{a})X + \wedge^2 X.$$

– obvious notion of minimal models.

Example: ① $R[X]$ is a minimal model for $R/(\underline{g})$

(②) $R[X,Y]$ is a minimal model for $R/(\underline{g})$

$\Leftrightarrow$ $(\underline{f}) \subseteq (\underline{a})(\underline{g})$.

Fact: • Minimal models ~~exist~~ exist and are unique up to isomorphism of dg algebras.

Remark: A Minimal model $(R[X], \partial)$ of $R/_I$ is a free resolution, but it is rarely a minimal free resolution.

– The minimal free resln. of $R/_I$ is a direct summand of $R[X]$.

(as $\subset$ (X).

Now fix minimal models $\begin{cases} S[X] & \text{of the } S\text{-algebra } R \\ R[Y] & \text{of the } R\text{-algebra } k. \end{cases}$

$$\begin{array}{ccc} & & S[X] \\ & \nearrow & \downarrow \simeq \\ S & \twoheadrightarrow & R \end{array} \qquad \begin{array}{ccc} & & R[Y] \\ & \nearrow & \downarrow \simeq \\ R & \twoheadrightarrow & k \end{array}$$

Fact: (1) (Gulliksen, Schöller) When char $k = 0$, $R[Y]$ is a minimal free resolution: $\partial(Y) \subseteq (\underline{a})\, R[Y]$

- [There is a statement in all characteristics, but we need ~~models~~ models with divided powers.]

(2) Avramov: There is a bijection

$$X_n \longleftrightarrow Y_{n+1} \qquad \forall \quad n \geq 1$$

- (even as graded sets)

In particular, the algebras $S[X]$ and $R[Y]$ determine each other.

(IV) Now suppose R is Koszul.

- Then $\forall\ y \in Y_{n+1}$ $\deg(y) = n+1$. (by Gulliksen/Schöller)

Thus, $\forall\ x \in X_{n+1}$ $\deg(x) = n+1$ (by Avramov)

[In fact, more is true: ∂X is purely quadratic, and can be recovered from ∂Y]

Now $\mathrm{Tor}_i^S(k,R) = H_i(k \otimes_S S[X])$

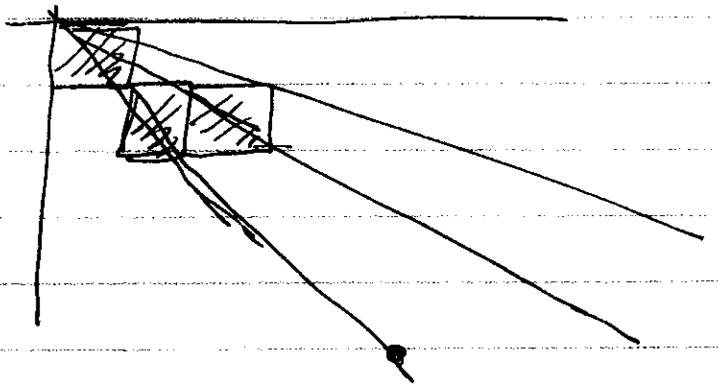
$= H_i(k[X])$

$\therefore \ \mathrm{top}\ \mathrm{Tor}_i^S(k,R) \le \mathrm{top}\,(k[X])_i \le 2i$

by previous observation + computation.

$2 \in X_1$

$\deg X_1 = 2$



$n(2,3)$

– Explain: $k[X_1] \otimes k[X_2] \otimes k[X_3] \cdots$

References:

① L. Avramov, Infinite free resolutions,

② L. Avramov, A. Conca, S.B. Iyengar, Free resolutions over commutative Koszul algebras