Simplicial Complexes Associated to a Relation
A Theorem of Dowker

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Certificate

This is to certify that the Dissertation entitled *Simplicial Complexes Associated to a Relation - A Theorem of Dowker* is a bonafide record of work done by Saurabh Trivedi under my supervision.

This Dissertation is to be submitted to Chennai Mathematical Institute in partial fulfillment of the requirements of the M.Sc. degree.

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Dated

Introduction

One of the ways in which cohomology groups of a topological space can be defined is to approximate the space by simplicial complexes and define the cohomology of the space as the "limit" of the cohomology groups of these simplicial complexes. This was achieved by *E. Čech.* If $\mathcal{A}$ is an open covering of a topological space X, one defines a simplicial complex $N(\mathcal{A})$, called its "nerve" and if $\mathcal{A}$ runs over all the open coverings of X, one obtains a direct system (under suitable conditions) of the cohomology groups of $N(\mathcal{A})$'s. The limit group of this direct system is called *Čech cohomology* groups of the space X.

A topological space can also be approximated by the simplicial complexes built of "small" simplices of finite set of points of the space. If $\mathcal{A}$ is an open covering of a space X, one defines its "Vietoris complex", $V(\mathcal{A})$, and if $\mathcal{A}$ runs over all the open coverings of X, one obtains a direct system (under suitable conditions) of the cohomology groups of $V(\mathcal{A})$'s. The limit group of this direct system is called *Alexander cohomology* groups of the space X. The construction of the nerve and Vietoris complex are in some sense notions dual to each other as we will see.

C. H. Dowker, in his paper "Homology groups of Relations" (Ann. of Math., 1952) associated two simplicial complexes with respect to a relation between two sets and proved that the homology and cohomology groups of the two simplicial complexes are isomorphic. In particular, we can define a relation between the points of a space X and the elements of a covering $\mathcal{A}$ as follows:

An element $x \in X$ and $U \in \mathcal{A}$ are related if and only if $x \in U$.

The two complexes associated to this relation are nerve and Vietoris complex of the covering $\mathcal{A}$. The nerve and the Vietoris complex are used to define the Čech and Alexander cohomology groups of X. Dowker has proved as an application of his theorem that these cohomology groups are isomorphic. We give an exposition of the proof here.

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Contents

Chapter 1

Preliminaries

1 Chain complexes

Definition 1.1 *A* graded (abelian) group *G is a collection of abelian groups $\{G_i\}$ indexed by integers with component-wise operations. Let G and G' be two graded groups; a homomorphism $f : G \to G'$ of graded groups of degree r is a collection of homomorphisms $\{f_i\}$,*

$$f_i : G_i \to G'_{i+r}.$$

A subgroup H of a graded group G is a graded group $\{H_i\}$ where H_i is a subgroup of G_i. The quotient group G/H is the graded group $\{G_i/H_i\}$.

Definition 1.2 *A* chain complex *$\{C, \partial\}$ is a graded group $C = \{C_i\}$ along with a homomorphism*

$$\partial : C \to C$$

of degree -1 such that $\partial \circ \partial = 0$. Equivalently, a chain complex is a sequence of abelian groups $\{C_i\}$ and homomorphisms $\{\partial_i\}$,

$$\cdots \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-1}} \cdots$$

such that $\partial_{i-1} \circ \partial_i = 0$ for each i. The homomorphism ∂ is called the boundary operator. *Let $\{C, \partial\}$ and $\{C', \partial'\}$ be two chain complexes; a homomorphism*

$\Phi : C \to C'$ of degree 0 is called a chain map *if for each i, the diagram*

$$\begin{array}{ccc} C_i & \xrightarrow{\partial_i} & C_{i-1} \\ {\scriptstyle \Phi_i}\downarrow & & \downarrow{\scriptstyle \Phi_{i-1}} \\ C'_i & \xrightarrow{\partial'_i} & C'_{i-1} \end{array}$$

is commutative.

In general, a chain map need not have degree 0 by definition, but for our purposes we assume that chain maps have degree 0.

Let $\{C, \partial\}$ be a chain complex. Since $\partial_i \circ \partial_{i+1} = 0$ for each i, $\text{Im } \partial_{i+1} \subseteq \ker \partial_i$, and so we can form the quotient group $\ker \partial_i / \text{Im } \partial_{i+1}$. We denote by $Z_i(C)$ and $B_i(C)$ the kernel of ∂_i and image of ∂_{i+1} respectively. The elements of $Z_i(C)$ are called 'cycles' and the elements of $B_i(C)$ are called 'boundaries'.

Definition 1.3 *Let $\{C, \partial\}$ be a chain complex. The* homology group *of $\{C, \partial\}$ is the graded group $H_*(C) = \{H_i(C)\} = \{Z_i(C)/B_i(C)\}$.*

If Φ is a chain map between two complexes $\{C, \partial\}$ and $\{C', \partial,\}$ then for each i, $\Phi(Z_i(C)) \subseteq \Phi(Z_i(C'))$ and $\Phi(B_i(C)) \subseteq \Phi(B_i(C'))$. Therefore, Φ induces a homomorphism $\Phi_* : H_*(C) \to H_*(C')$ of degree 0 on the homology groups of the two chain complexes.

Suppose $\{C, \partial\}$ and $\{C', \partial'\}$ are two chain complexes and $T : C \to C'$ a homomorphism of graded groups of degree 1. We have the following diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{i+1} & \xrightarrow{\partial_{i+1}} & C_i & \xrightarrow{\partial_i} & C_{i-1} & \longrightarrow & \cdots \\ & & & {\scriptstyle T_{i+1}}\swarrow & & {\scriptstyle T_i}\swarrow & & & \\ \cdots & \longrightarrow & C'_{i+1} & \xrightarrow{\partial'_{i+1}} & C'_i & \xrightarrow{\partial'_i} & C'_{i-1} & \longrightarrow & \cdots \end{array}$$

Consider the homomorphism $\partial' T + T\partial : C \to C'$ of degree 0. This is a chain map because

$$\partial'(\partial' T + T\partial) = \partial'\partial' T + \partial' T\partial = \partial' T\partial = \partial' T\partial + T\partial\partial = (\partial' T + T\partial)\partial.$$

Now if $z \in Z_i(C)$, $(\partial' T + T\partial)(z) = \partial' T(z)$ which is in $B_i(C')$. Thus, $(\partial' T + T\partial)_*$ is a zero homomorphism. We make the following definition:

Definition 1.4 *Two chain maps $f, g : C \to C'$ are called* chain homotopic *if there exists a homomorphism $T : C \to C'$ of degree 1 such that $\partial' T + T\partial = f - g$. T is then called a* chain homotopy *between f and g.*

Proposition 1.5 *If $f, g : C \to C'$ are chain homotopic chain maps, then they induce same homomorphism on homology groups. That is, $f_* = g_*$ as homomorphisms from $H_*(C)$ to $H_*(C')$.*

Proof: Let $T : C \to C'$ be a chain homotopy between f and g, then

$$f_* - g_* = (f - g)_* = (\partial' T + T\partial)_* = 0.$$

and so $f_* = g_*$. □

Definition 1.6 *Let $\{C, \partial\}$ and $\{C', \partial'\}$ be two chain complexes. A chain map $\phi : C \to C'$ is called a* chain equivalence *if there is a chain map $\phi' : C' \to C$ such that $\phi' \circ \phi$ and $\phi \circ \phi'$ are chain homotopic to the identity maps of C and C', respectively. We call ϕ' a* chain homotopy inverse *to ϕ.*

Let $\{C, \partial\}$ be a chain complex. $\{D, \partial\}$ is a *subcomplex* of $\{C, \partial\}$ if $D_i \subseteq C_i$ is a subgroup for each i and the boundary operator for D is the restriction of the boundary operator for C. Define the quotient chain complex $C/D = \{C_i/D_i, \partial'\}$, where $\partial'([c]) = [\partial(c)]$ for the coset containing c.

Definition 1.7 *Let $\{C, \partial\}$ be a chain complex and $\{D, \partial\}$ be a subcomplex. The homology group $H_*(C/D)$ is called the* relative homology *group of C mod D.*

Definition 1.8 *A cochain complex $\{C, \delta\}$ is a graded group $C = \{C^i\}$ along with a homomorphism*

$$\delta : C \to C$$

of degree 1 such that $\delta \circ \delta = 0$. Equivalently, a cochain complex is a sequence of abelian groups $\{C^i\}$ and homomorphisms $\{\delta_i\}$,

$$\cdots \xrightarrow{\delta_{i-1}} C^i \xrightarrow{\delta_i} C^{i+1} \xrightarrow{\delta_{i+1}} \cdots$$

such that $\delta_i \circ \delta_{i-1} = 0$ *for each* i*. The homomorphism* δ *is called the* coboundary operator*. Let* $\{C, \delta\}$ *and* $\{D, \delta'\}$ *be two cochain complexes, a homomorphism* $\Phi : C \to D$ *of degree* 0 *is called a* cochain map *if for each* i*, the diagram*

$$\begin{array}{ccc} C^i & \xrightarrow{\delta_i} & C^{i+1} \\ \Big\downarrow{\scriptstyle \Phi_i} & & \Big\downarrow{\scriptstyle \Phi_{i+1}} \\ D^i & \xrightarrow{\delta'_i} & D^{i-1} \end{array}$$

is commutative.

In general, a cochain map need not have degree 0 by definition, but for our purposes we assume that cochain maps have degree 0.

Let $\{C, \delta\}$ be a cochain complex. Since $\delta_i \circ \delta_{i-1} = 0$ for each i, Im $\delta_{i-1} \subseteq \ker \delta_i$, and so we can form the quotient group $\ker \delta_i/\text{Im } \delta_{i-1}$. We denote by $Z^i(C)$ and $B^i(C)$ the kernel of δ_i and image of δ_{i-1} respectively. The elements of $Z^i(C)$ are called 'cocycles' and the elements of $B^i(C)$ are called 'coboundaries'.

Definition 1.9 *Let* $\{C, \partial\}$ *be a cochain complex. The* cohomology group *of* $\{C, \delta\}$ *is the graded group* $H^*(C) = \{H^i(C)\} = \{Z^i(C)/B^i(C)\}$*.*

If Φ is a cochain map between two complexes $\{C, \delta\}$ and $\{D, \delta'\}$, then for each i, $\Phi(Z^i(C)) \subseteq \Phi(Z^i(C'))$ and $\Phi(B^i(C)) \subseteq \Phi(B^i(C'))$. Therefore, Φ induces a homomorphism $\Phi^* : H^*(C) \to H^*(D)$ of degree 0 on the cohomology groups of the two cochain complexes.

Suppose $\{C, \delta\}$ and $\{D, \delta'\}$ are two cochain complexes and $T : C \to D$ is a homomorphism of graded groups of degree -1. We have the following diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C^{i-1} & \xrightarrow{\delta_{i-1}} & C^i & \xrightarrow{\delta_i} & C^{i+1} & \longrightarrow & \cdots \\ & & & \swarrow{\scriptstyle T_i} & & \swarrow{\scriptstyle T_{i+1}} & & & \\ \cdots & \longrightarrow & D^{i-1} & \xrightarrow{\delta'_{i-1}} & D^i & \xrightarrow{\delta'_i} & D^{i+1} & \longrightarrow & \cdots \end{array}$$

Consider the homomorphism $\delta'T + T\delta : C \to D$ of degree 0. This is a cochain map because

$$\delta'(\delta'T + T\delta) = \delta'\delta'T + \delta'T\delta = \delta'T\delta = \delta'T\delta + T\delta\delta = (\delta'T + T\delta)\delta.$$

Now if $z \in Z^i(C)$, $(\delta' T + T\delta)(z) = \delta' T(z)$ which is in $B^i(C')$. Thus, $(\delta' T + T\delta)^*$ is a zero homomorphism. We make the following definition:

Definition 1.10 *Two cochain maps $f, g : C \to D$ are called* cochain homotopic *if there exists a homomorphism $T : C \to D$ of degree -1 such that $\delta' T + T\delta = f - g$. T is then called a* cochain homotopy *between f and g.*

Proposition 1.11 *If $f, g : C \to D$ are cochain homotopic cochain maps, then they induce same homomorphism on cohomology groups. That is, $f^* = g^*$ as homomorphisms from $H^*(C)$ to $H^*(D)$.*

Proof: Let $T : C \to D$ be a cochain homotopy between f and g, then

$$f^* - g^* = (f - g)^* = (\delta' T + T\delta)^* = 0.$$

and so $f^* = g^*$. □

Definition 1.12 *Let $\{C, \delta\}$ and $\{D, \delta'\}$ be two cochain complexes. A cochain map $\phi : C \to D$ is called a* cochain equivalence *if there is a cochain map $\phi' : D \to C$ such that $\phi' \circ \phi$ and $\phi \circ \phi'$ are cochain homotopic to the identity maps of C and D, respectively. We call ϕ' a* cochain homotopy inverse *to ϕ.*

Let $\{C, \delta\}$ be a cochain complex. $\{D, \delta\}$ is a *subcomplex* of $\{C, \delta\}$ if $D^i \subseteq C^i$ is a subgroup for each i and the coboundary operator for D is the restriction of the coboundary operator for C. Define the quotient cochain complex $C/D = \{C^i/D^i, \delta'\}$, where $\delta'([c]) = [\partial(c)]$ for the coset containing c.

Definition 1.13 *Let $\{C, \delta\}$ be a cochain complex and $\{D, \delta\}$ be a subcomplex. The cohomology group $H^*(C/D)$ is called the* relative cohomology *group of C mod D.*

Note that if $\{C, \delta\}$ is a cochain complex and we define $D_i = C^{-i}$ and $\partial = \delta : D_i \to D_{i-1}$, then $\{D, \partial\}$ becomes a chain complex.

Let A and G are abelian groups and $\mathrm{Hom}(A, G)$ be the abelian group of all homomorphisms from A to G. Any homomorphism $\phi : A \to B$ induces a homomorphism

$$\phi^{\#} : \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$$

defined by $\phi^{\#}(f) = f \circ \phi$. Observe that $(\psi \circ \phi)^{\#} = \phi^{\#} \circ \psi^{\#}$.

If $A = \{A_i\}$ is a graded group and G is any abelian group then

$$\mathrm{Hom}(A, G) = \{\mathrm{Hom}(A_i, G)\}$$

is also a graded group. If $A = \{A_i\}$ and $B = \{B_i\}$ are graded groups and $\phi : A \to B$ is a homomorphism of degree r then $\phi^{\#} = \{\phi_i^{\#}\}$, where

$$\phi_i^{\#} : \mathrm{Hom}(B_{i+r}, G) \to \mathrm{Hom}(A_i, G)$$

, is a homomorphism from the graded group $\mathrm{Hom}(B, G)$ to $\mathrm{Hom}(A, G)$ of degree $-r$.

Now suppose $\{C, \partial\}$ is a chain complex and G is an abelian group, then $\mathrm{Hom}(C, G) = \{C^i\} = \{\mathrm{Hom}(C_i, G)\}$ along with $\delta = \{\delta_i\}$, where $\delta_i = \partial_{i+1}^{\#} : C^i \to C^{i+1}$ is a cochain complex, for

$$\delta \circ \delta = \partial^{\#} \circ \partial^{\#} = (\partial \circ \partial)^{\#} = 0.$$

Therefore $\{\mathrm{Hom}(C, G), \delta\}$ is a cochain complex.

If $\{C, \partial\}$ and $\{D, \partial'\}$ are two chain complexes and $f : C \to D$ is chain map, then for any abelian group G

$$f^{\#} : \mathrm{Hom}(D, G) \to \mathrm{Hom}(C, G)$$

is a cochain map. Further, if f, g are chain homotopic then $f^{\#}, g^{\#}$ are cochain homotopic.

2 Simplicial Complexes

Definition 2.1 *A set* $\{a_i\}_{i=0}^n$ *of points in* $\mathbb{R}^N$ *is said to be* geometrically independent *if for any real numbers* $\{t_i\}_{i=0}^n$*, the equations*

$$\sum_{i=0}^{n} t_i = 0 \ \ \textit{and} \ \ \sum_{i=0}^{n} t_i a_i$$

imply that $t_i = 0$ *for each* i*. The* n-plane P *spanned by this geometrically independent set is the set of all points* x *of* $\mathbb{R}^N$ *such that*

$$x = \sum_{i=0}^{n} t_i a_i,$$

for some real numbers $\{t_i\}_{i=0}^n$ *with* $\sum t_i = 1$*.*

Observe that, saying $\{a_i\}_{i=0}^n$ is a set of geometrically independent points is same as saying that the set of vectors $\{a_i - a_0\}_{i=1}^n$ are linearly independent.

A one-point set, two distinct points in a line, three non-colinear points in a plane and four non-planar points in $\mathbb{R}^3$ are the examples of geometrically independent sets.

Definition 2.2 *Let* $\{a_i\}_{i=0}^n$ *be a geometrically independent set in* $\mathbb{R}^N$*. The* n-simplex σ *spanned by* $a_0, ..., a_n$ *is the set of all points* x *of* $\mathbb{R}^N$ *such that*

$$x = \sum_{i=0}^{n} t_i a_i, \quad \textit{where} \quad \sum_{i=0}^{n} t_i = 1$$

and $t_i \geq 0$ *for all* i*. The numbers* t_i *are uniquely determined by* x*; they are called the barycentric coordinates of the point* x *and denoted by* $t_i(x)$*.*

A 0-simplex is a point, a 1-simplex spanned by a_0 and a_1 is the line segment joining them, a 2-simplex spanned by a_0, a_1, a_2 (geometrically independent) is the triangle having these three points as vertices and a 3-simplex spanned by a_0, a_1, a_2, a_3 (geometrically independent) is the tetrahedron having these four points as vertices.

Let $\{a_i\}_{i=0}^n$ be a geometrically independent set and σ be the n-simplex spanned by this set. The points $a_0, ..., a_n$ are called the *vertices* of σ and n is called the *dimension* of σ and is denoted by $\dim \sigma$. Any simplex spanned by a subset of $\{a_i\}_{i=0}^n$ is called a *face* of σ.σ itself is a face of σ and the faces of σ different from σ are called the *proper faces* of σ. The union of all proper faces is called the *boundary* of σ denoted by $\operatorname{Bd} \sigma$. The *interior* of σ, $\operatorname{Int} \sigma$, is the set of all points of σ which do not lie in the $\operatorname{Bd} \sigma$. It follows that at least one of the barycentric coordinates $t_i(x)$ of all the points x lying in the $\operatorname{Bd} \sigma$ is 0 and $t_i(x) > 0$ $(i = 0, 1, ..., n)$ for all points x lying in the $\operatorname{Int} \sigma$.

We list some properties of n-simplices.

1. The barycentric coordinates $t_i(x)$ are continuous functions of x.

2. σ is the union of all the line segments joining a_0 to the points of the simplex s spanned by $a_1, ..., a_n$. Two such line segments intersect only in the point a_0.

3. σ is a compact, convex set in $\mathbb{R}^N$, which is the intersection of all convex sets in $\mathbb{R}^n$ containing $a_0, ..., a_n$.

4. $\operatorname{Int}\sigma$ is convex and is open in the plane P; its closure is σ.

Definition 2.3 *A* simplicial complex K *in* $\mathbb{R}^N$ *is a collection of simplices in* $\mathbb{R}^N$ *such that:*

(1) Every face of a simplex of K *is in* K*.*

(2) The intersection of any two simplices of K *is a face of each of them.*

The following lemma is sometimes useful in verifying that a collection of simplices is a simplicial complex.

Lemma 2.4 *A collection* K *of simplices is a simplicial complex if and only if the following hold:*

(1) Every face of a simplex of K *is in* K*.*

(2) Every pair of distinct simplices of K *have disjoint interior.*

Proof: Assume K is a simplicial complex and σ and τ be two simplices of K with $x \in \operatorname{Int}\sigma \cap \operatorname{Int}\tau$. Suppose $s = \sigma \cap \tau$. We know that s is a face of each of σ and τ. If s were a proper face of σ then x would belong to Bd σ, a contradiction. Therefore $s = \sigma$. Similarly $s = \tau$. This imply that $\sigma = \tau$.

Now suppose K is a collection of simplices satisfying (1) and (2). We show that if $\sigma \cap \tau \neq \emptyset$, then $\sigma \cap \tau$ is the face σ' of σ that is spanned by those vertices $b_0, ..., b_n$ that lie in τ. Since $\sigma \cap \tau$ is convex $\sigma' \subset \sigma \cap \tau$. Now, suppose $x \in \sigma \cap \tau$. Then $x \in \operatorname{Int} s \cap \operatorname{Int} t$, for some face s of σ and some face t of τ. From (2), we have $s = t$; hence the vertices of s lie in τ, so that by definition they are elements of the set $\{b_0, ..., b_m\}$. Then s is a face of σ', so that $s \in \sigma'$

as desired. ◻

If σ is a simplex, then the collection consisting of σ and all its proper faces form a simplicial complex since for each point $x \in \sigma$, there is exactly one face s of σ with $x \in \operatorname{Int} \sigma$.

Definition 2.5 *If L is a subcollection of a simplicial complex K that contains all faces of its elements, then L itself is a simplicial complex. Such a subcollection is called a* subcomplex *of K. The p-*skeleton $K^{(p)}$ *of K is the subcomplex consisting of all simplices of K of dimension at most p. The points of the subcomplex $K^{(0)}$ are called the* vertices *of K.*

Let K be a simplicial complex. Let $|K|$ be the subset of $\mathbb{R}^N$ that is the union of the simplices of K. There are two ways in which we can topologize $|K|$.

(1). By giving subspace topology as a subspace of $\mathbb{R}^N$.

(2). By giving each simplex its natural topology and then declaring a subset A of $|K|$ to be closed if and only if $A \cap \sigma$ is closed in σ for each σ in K.

In general, the topology given by (2) is finer than the topology given by (1).

Definition 2.6 $|K|$ *with the topology given by (2) is called the* polytope *of K.*

We will not go into more details of the topology of a simplicial complex. We simply state the following facts, which we need later, about the polytope of a simplicial complex.

Let K be a simplicial complex and $|K|$ be its polytope then

1. If L is a subcomplex of K, then $|L|$ is a closed subspace of $|K|$. In particular, if $\sigma \in K$, then σ is a closed subspace of $|K|$.
2. A map $f : |K| \to X$ is continuous if and only if $f|_\sigma$ is continuous for each $\sigma \in K$.

3. $|K|$ is Hausdorff.

4. If K is finite, then $|K|$ is compact. Conversely, if a subset A of $|K|$ is compact, then $A \in |K_0|$ for some finite subcomplex K_0 of K.

5. The complex K is locally finite (each vertex of K belongs only to finitely many simplices of K) if and only if $|K|$ is locally compact.

3 Simplicial maps

Definition 3.1 *Let K and L be two simplicial complexes. A map $f : K \to L$ from the vertex set of K to the vertex set of L is called a* simplicial map *if whenever $\{a_i\} \subseteq K^{(0)}$ span a simplex of K, $\{f(a_i)\} \subseteq L^{(0)}$ span a simplex of L. f is called an isomorphism if it is bijective and $\{v_0, ..., v_n\}$ span a simplex of K if and only if $\{f(v_0), ..., f(v_n)\}$ span a simplex of L. Two simplicial comlpexes are called isomorphic if there is an isomorphism between them.*

Lemma 3.2 *Let K and L be complexes, and let $f : K \to L$ be a simplicial map. Then f can be extended to a continuous map $\overline{f} : |K| \to |L|$ from the polytope of K to polytope of L such that $\overline{f}|_{K^{(0)}} = f$ and*

$$x = \sum_{i=0}^{n} t_i v_i \quad \Rightarrow \quad \overline{f}(x) = \sum_{i=0}^{n} t_i f(v_i)$$

for any set of vertices $\{v_0, ..., v_n\}$ which span a simplex of K. Further, if f is an isomorphism then $\overline{f}$ is a homeomorphism.

Proof: Since f is a simplical map, for any set of vertices $\{v_0, ..., v_n\}$ which span a simplex σ of K, $\{f(v_0), ..., f(v_n)\}$ span a simplex τ of L even if $f(v_i)$'s are not distinct, therefore if $x \in \sigma$ such that

$$x = \sum_{i=0}^{n} t_i v_i \text{ for some } t_i \in \mathbb{R} \text{ then } \sum_{i=0}^{n} t_i f(v_i) \in \tau$$

therefore, f can be extended to $|K|$ in the obvious way by defining

$$\overline{f}(x) = \overline{f}(\sum_{i=0}^{n} t_i v_i) = \sum_{i=0}^{n} t_i f(v_i).$$

It is clear by the defintion of $\overline{f}$ that $\overline{f}|_{K^{(0)}} = f$. The map $\overline{f}$ is continuous as a map of σ into τ, and hence as a map of σ into $|L|$. Then by the property (2) listed above, $\overline{g}$ is continuous as a map of $|K|$ into $|L|$.

Now suppose f is an isomorphism. Since f is a bijection, $\overline{f}$ maps each simplex σ of K onto a simplex τ of L of the same dimension as σ. We need only to show that the map $\overline{f^{-1}} : \tau \to \sigma$ induced by f^{-1} is the inverse of the map $\overline{f} : \sigma \to \tau$. If $x = \sum t_i v_i$, then $\overline{f}(x) = \sum t_i f(v_i)$; whence

$$\overline{f^{-1}}(\overline{f}(x)) = \overline{f^{-1}}(\sum t_i f(v_i)) = \sum t_i f^{-1}(f(v_i)) = \sum t_i v_i = x.$$

□

Corollary 3.3 *Let Δ^N denote the simplicial complex consisting of an N-simplex and its faces. If K is finite simplicial complex, then K is isomorphic to some subcomplex of Δ^N .*

Proof: Let $v_0, ..., v_n$ be the vertices of K. Choose $a_0, ..., a_N$ to be geometrically independent points in $\mathbb{R}^N$, and let Δ^N consist of the N-simplex they span, along with its faces. The simplicial map $f : K^{(0)} \to \{a_0, ..., a_N\}$ defined by $f(v_i) = a_i$ is an isomorphism onto a subcomplex of Δ^N. □

Till now we have assumed that a simplicial complex must lie in $\mathbb{R}^N$ and this puts a restriction on the dimension of the simplices of the simplicial complex, namely the dimension of a simplex of a simplicial complex lying in $\mathbb{R}^N$ can at most be equal to N. We can remove this restriction by assuming that our simplicial complex lies in the generalized euclidean space E^J which is defined as follows:

Let J be an arbitrary index set and let $\mathbb{R}^J$ be the $J - fold$ product of $\mathbb{R}$. Let E^J denote the subset of $\mathbb{R}^J$ of all points $(x_\alpha)_{\alpha \in J}$ such that $x_\alpha = 0$ for all but finitely many values of α. Then E^J is a vector space. If ϵ_α is the map of J into $\mathbb{R}$ whose value is 1 on the index α and 0 on all other elements of J, then $\{\epsilon_\alpha \,|\, \alpha \in J\}$ is a basis of E^J. We topologize it by the metric $|x - y| = \max\{|x_\alpha - y_\alpha| \,|\, \alpha \in J\}$.

Assuming the the simplicial complex lies in E^J, we say that the dimension of the simplicial complex is n if the largest dimension of its simplices is n. If there is no such n we say that the simplicial complex is infinite dimensional.

We define an abstract simplicial complex as follows:

Definition 3.4 *An* abstract simplicial complex K *consists of a set* $V = \{v\}$ *of* vertices *and a set* $\{s\}$ *of finite non-empty subsets of* V *called* simplices *such that*

(1). Any set with exactly one vertex is a simplex.

(2). If s *is a simplex then every nonempty subset of* s *is a simplex.*

The dimension of a simplex is one less than the number of vertices in it. A non-empty subset s' of a simplex s is called a face of s. Let K be an abstract simplicial complex, a subcomplex of K is a subcollection of its simplices which itself is a complex.

The definitions of simplicial map and isomorphism for abstract simplicial complexes remain the same as definition 3.1.

Definition 3.5 *Let* K *be a simplicial complex and* V *be the vertex set of* K*. Let* $\mathcal{K}$ *be the collection of all subsets of* V *which span a simplex of* K*. The collection* $\mathcal{K}$ *is called the vertex scheme of* K*.*

Vertex scheme of a simplicial complex is an example of an abstract simplicial complex. We in fact have the following theorem:

Theorem 3.6 *Every abstract simplicial complex* K *is isomorphic to the vertex scheme of some simplicial complex* L*.*

Proof: Given an index set J, let Δ^J be the collection of all simplices in E^J spanned by finite subsets of the standard basis $\{e_\alpha\}$ for E^J. Δ^J is a simplicial complex for if σ and τ are two simplices of Δ^J, then their combined vertex set is geometrically independent and spans a simplex of Δ^J. We shall call Δ^J an "infinite-dimensional simplex".

Now let K be an abstract simplicial complex with vertex set V. Choose an index set J large enough that there is an injective map $f : V \to \{e_\alpha\}_{\alpha \in J}$. Let L be the subcomplex of Δ^J consisting of the simplices spanned by $\{f(a_0), ..., f(a_n)\}$ for any simplex of K spanned by $\{a_0, ..., a_n\}$. Then clearly,

L is a simplicial complex and K is isomorphic to vertex scheme of L. $\square$

If the abstract simplicial complex K is isomorphic to the vertex scheme of the simplicial complex L, we call L a geometric realization of K.

4 Homology of Simplicial complexes

We now associate with each simplicial complex certain graded groups called the homology group and cohomology group of the simplicial complexes.

Definition 4.1 *Let σ be an n-simplex. We say that two orderings of its vertices are equivalent if they differ from one another by an even permutation. If dim $\sigma > 0$, the orderings of the vertices of σ fall into two equivalence classes. These classes are called the* orientations *of σ. An oriented simplex is a simplex σ together with an orientation of σ.*

Note that if σ is a 0-simplex then there is only one orientation of σ.

If the vertices of σ are $\{v_0, ..., v_n\}$ then we denote the oriented simplex σ by $[v_0, ...v_n]$ for the particular ordering $(v_0, ..., v_n)$. Also for convenience, we denote by σ and σ' the two opposite orientations of σ.

The following is true for both geometric and abstract simplicial complexes so from now on by a simplicial complex we mean either geometric or abstract simplicial complex. We distinguish between them whenever necessary.

Definition 4.2 *Let K be a simplicial complex. Any function c from the set of oriented p-simplices of K to the integers satisfying:*

(1). $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same simplex σ.

(2). $c(\sigma) = 0$ for all but finitely many oriented p-simplices σ.
is called a p-chain on K.

We can add p-chains by adding their values and this puts a groups structure on the set of all p-chains on K. We denote that group by $C_p(K)$.

If σ is an oriented p-simplex of K, the function c defined by: $c(\sigma) = 1$

$c(\sigma') = -1$

$c(\tau) = 0$ if $\tau \neq \sigma$ or σ'
is called the elementary p-chain corresponding to σ.

By abuse of notation, we use the symbol σ to denote not only a simplex, but also the oriented simplex and the elementary p-chain corresponding to σ. With this notation $\sigma' = -\sigma$.

It is easily seen that $C_p(K)$ is nothing but the abelian group generated by the set of all oriented p-simplices of K with the relation $\sigma + \sigma' = 0$, we use the corresponding elementary chains as the basis for $C_p(K)$. That is, any element of $C_p(K)$ or any p-chain can be uniquely written as a linear combination $c = \sum n_i \sigma_i$ where n_i is 0 for all but finitely many i's. c assigns the value n_i to oriented p-simplex σ_i, the value $-n_i$ to σ_i' and the value 0 for all other oriented p-simplices.

For $p < 0$ we take $C_p(K)$ to be the trivial group.

Now, let

$$\partial_p : C_p(K) \to C_{p-1}(K)$$

be the homomorphism got by extending the following map defined on the basis of $C_p(K)$ by

$$\partial_p(\sigma) = \partial_p([v_0, ..., v_p]) = \sum_{i=0}^{p} (-1)^i [v_0, ..., \hat{v}_i, ..., v_p]$$

where σ is the oriented simplex $[v_0, ..., v_p]$ and the symbol $\hat{v}_i$ means that the vertex v_i is deleted from the ordering of the simplex. Since $C_p(K)$ is the trivial group for $p < 0$, the homomorphism ∂_p is the trivial homomorphism for $p \leq 0$.

Lemma 4.3 *∂_p is well defined and $\partial_p(-\sigma) = -\partial(\sigma)$.*

Proof: It suffices to show that the right hand side of the above equation changes sign if we exchange two adjacent vertices in the array $[v_0, ..., v_p]$. We compare the expressions for

$$\partial_p([v_0, ..., v_j, v_{j+1}, ..., v_p]) \;\; and \;\; \partial_p([v_0, ..., v_{j+1}, v_j, ..., v_p]).$$

For $i \neq j$, the ith terms in the two expressions differ precisely by a sign; the terms are identical except that v_j and v_{j+1} have been interchanged. For $i = j$ and $i = j + 1$ we have

$$(-1)^J[..., v_{j-1}, \hat{v}_j, v_{j+1}, v_{j+2}, ...] + (-1)^{j+1}[..., v_{j-1}, v_j, \hat{v}_{j+1}, v_{j+2}, ...]$$

and

$$(-1)^J[..., v_{j-1}, \hat{v}_{j+1}, v_j, v_{j+2}, ...] + (-1)^{j+1}[..., v_{j-1}, v_{j+1}, \hat{v}_j, v_{j+2}, ...].$$

Comparing, we see that these two expressions differ a sign. □

Lemma 4.4 $\partial_{p-1} \circ \partial_p = 0$ *for all* p.

Proof: We compute

$$\begin{aligned} \partial_{p-1}\partial_p([v_0, ..., v_p]) &= \sum_{i-0}^{p} (-1)^i \partial_{p-1}([v_0, ..., \check{v}_i, ..., v_p]) \\ &= \sum_{j<i} (-1)^i (-1)^j [..., \check{v}_j, ..., \check{v}_i, ...] \\ &\quad + \sum_{j>i} (-1)^i (-1)^{j-1} [..., \check{v}_i, ..., \check{v}_j, ...] \end{aligned}$$

All the terms in the summation cancel in pairs which gives $\partial_{p-1} \circ \partial_p = 0$. □

By the above discussion we conclude that $\{C(K), \partial\}$, where $C(K) = \{C_p(K)\}$ and $\partial = \{\partial_p\}$ is a chain complex. $C(K)$ is sometimes called the chain group of K.

Let G be an abelian group. Since $\{C(K), \partial\}$ is a chain complex,

$$\{\mathrm{Hom}(C(K), G), \delta\}$$

where

$$\mathrm{Hom}(C(K), G) = \{\mathrm{Hom}(C_p(K), G)\} \text{ and } \delta = \partial^{\#} = \{\partial_p^{\#}\}$$

is a cochain complex.

Definition 4.5 *Let K be a simplicial complex. The homology group of the chain complex $\{C(K), \partial\}$ is called the homology group of the simplicial complex K. We denote this group by $H_*(K)$.*

Definition 4.6 *Let K be a simplicial complex and G be an abelian group. The cohomology group of the cochain complex*

$$\{Hom(C(K), G), \delta\}$$

is called the cohomology group with coefficients in G of the simplicial complex K. We denote this group by $H^*(K, G)$.

Definition 4.7 *Let $\epsilon : C_0(K) \to \mathbb{Z}$ be the surjective homomorphism defined by $\epsilon(v) = 1$ for each vertex v of K. Then if c is a 0-chain, $\epsilon(c)$ equals the sum of the values of c on the vertices of K. The map ϵ is called an augmentation map for $C_0(K)$. If d is a 1-chain, then $\epsilon(\partial d) = 0$. We define the reduced homology group of K in dimension 0, denoted by $\hat{H}_0(K)$, by $\hat{H}_0(K) = ker\,\epsilon / im\,\partial_1$.*

Let K be a simplicial complex and K_0 be a subcomplex of K. For every p, $C_p(K_0)$ can be seen as a subgroup of $C_p(K)$ in the natural way. Therefore $C(K_0)$ is a subgroup of $C(K)$ as a graded group. The quotient graded group $C(K, K_0) = C(K)/C(K_0)$ is called the group of chains of K relative to K_0. This group with the boundary operator ∂, which is just the restriction of ∂ to $C(K_0)$ is a chain complex. The homology group $H_*(K, K_0)$ of $\{C(K, K_0), \partial\}$ is called the realtive homology group of K modulo K_0.

Definition 4.8 *(K, K_0) is called a pair.*

Chapter 2

Contiguity and Barycentric Subdivisions

In this chapter we will see that any simplicial map between simplicial complexes induces a homomorphism between the homology groups of the complexes. One is always interested in finding conditions under which two simplicial maps induce same homomorphism on homology and cohomology groups. We prove that contiguity of simplicial maps is one such condition under which it is possible. Later, we will define barycentric subdivisions of a simplicial complex and prove that the homology groups of simplicial complexes and homology groups of their subdivisions are isomorphic.

1 Homomorphism induced by simplicial maps

We recall that a simplicial map between two simplicial complexex K and L is a vertex map $f : K \to L$ such that whenever $\{v_0, ..., v_n\}$ span a simplex of K, $\{f(v_0), ..., f(v_n)\}$ span a simplex of L. We define

Definition 1.1 *Let $f : K \to L$ be a simplicial map. Define the homomorphism*

$$f_{p\#} : C_p(K) \to C_p(L)$$

by defining it on oriented simplices as follows:

$$f(x) = \begin{cases} [f(v_0), ..., f(v_p)] & \textit{if } f(v_0), ..., f(v_p) \textit{ are distinct} \\ 0 & \textit{otherwise} \end{cases}$$

This map is clearly well defined. The homomorphism

$$f_\# = \{f_{p\#}\} : C(K) \to C(L)$$

between the graded groups $C(K)$ and $C(L)$ is called the homomorphism induced by the simplicial map f.

Lemma 1.2 *$f_\# : C(K) \to C(L)$ is a chain map between the chain complexes $\{C(K), \partial\}$ and $\{C(L), \partial'\}$*

Proof: We need to show that for each p the diagram

$$\begin{array}{ccc} C_p(K) & \xrightarrow{\partial_p} & C_{p-1}(K) \\ \downarrow{\scriptstyle f_{p\#}} & & \downarrow{\scriptstyle f_{p-1\#}} \\ C_p(L) & \xrightarrow{\partial'_p} & C_{p-1}(L) \end{array}$$

is commutative. That is, we need to show that (we omit the subscripts from the homomorphisms for convenience) for each p

$$\partial' f_\#([v_0, ..., v_p]) = f_\#(\partial([v_0, ..., v_p]). \tag{2.1}$$

Let τ be the simplex of L spanned by $f(v_0), ..., f(v_p)$.
We have to consider the following cases depending on the dimension of τ.

Case 1. dim $\tau = p$.
In this case, the vertices $f(v_0), ..., f(v_p)$ are distinct and so

$$\begin{aligned} \partial' f([v_0, ..., v_p]) &= \partial'([f(v_0), ..., f(v_p)]) \\ &= \sum_{i=0}^{p} (-1)^i [f(v_0), ..., f(\check{v}_i), ..., f(v_p)] \\ &= f(\sum_{i=0}^{p} (-1)^i [v_0, ..., \check{v}_i, ..., v_p]) \\ &= f\partial([v_0, ..., v_p]) \end{aligned}$$

Case 2. dim $\tau \leq p-2$
In this case, the left hand side of (2.1) vanishes. The right side vanishes because at least two point of $f(v_0), ..., f(v_{i-1}), f(v_{i+1}), ..., f(v_p)$ are same.

Case3. dim $\tau = p-1$
In this case, we may assume that $f(v_0) = f(v_1)$ and $f(v_2), ..., f(v_p)$ are distinct. Then the left side of (2.1) vanishes. The right side equals

$$[f(v_1), f(v_2), ..., f(v_p)] - [f(v_0), f(v_2), ..., f(v_p)]$$

which vanishes since $f(v_0) = f(v_1)$. □

Since $f_\# : C(K) \to C(L)$ is a chain map, it induces a homomorphism $f_* : H_*(K) \to H_*(L)$ between the homology groups of K and L.

We have the following theorem:

Theorem 1.3 *(a). Let $i : K \to K$ be the identity simplicial map. Then $i_* : H_*(K) \to H_*(L)$ is the identity homomorphism.*

(b). Let $f : K \to L$ and $g : L \to M$ be simplicial maps. Then $(g \circ f)_* = g_* \circ f_*$.

Proof: It is immediate from the definition that $i_\#$ is the identity homomorphism and $(g \circ f)_\# = g_\# \circ f_\#$. The theorem follows. □

Lemma 1.4 *The chain map $f_\#$ preserves the augmentation map ϵ; therefore, it induces a homomorphism f_* of reduced homology groups.*

Proof: Let $f : K \to L$ be simplicial map. Then $\epsilon f_\#(v) = 1$ and $\epsilon(v) = 1$ for each vertex v of K. thus $\epsilon \circ f_\# = \epsilon$. This implies that $f_\#$ carries the kernel of $\epsilon_K : C_0(K) \to \mathbb{Z}$ into the kernel of $\epsilon_L : C_0(L) \to \mathbb{Z}$, and thus induces a homomorphism $f_* : \hat{H}_0(K) \to \hat{H}_0(L)$. □

2 Contiguity

We now define the notion of contiguity of two simplicial maps and prove that under the two contiguous simplicial maps induce the same homomorphism between the homology groups of simplicial complexes.

Definition 2.1 *Given two simplicial maps $f, g : K \to L$, these maps are said to be contiguous if for every set of points $v_0, ..., v_p$ which span a simplex of K, the points*

$$f(v_0), ..., f(v_p), g(v_0), ..., g(v_p)$$

span a simplex of L.

Before proving the theorem about contiguity we need to make the following definition.

Definition 2.2 *We say that a chain c is carried by a subcomplex L of K if c has the value 0 on every simplex that is not in L.*

Theorem 2.3 *If $f, g : K \to L$ are contiguous simplicial maps, then $f_* = g_*$.*

Proof: We know by proposition (1.11) of chapter 1, that to prove that $f_* = g_*$, it is enough to show that the two chain maps $f_\#$ and $g_\#$ are chain homotopic. Therefore we construct a chain homotopy between $f_\#$ and $g_\#$.

For each simplex $\sigma = v_0...v_p$ of K, let $L(\sigma)$ denote the subcomplex of L consisting of the simplex whose vertex set is $\{f(v_0), ..., f(v_p), g(v_0), ..., g(v_p)\}$, and its faces. We note the following facts:

(1). $L(\sigma)$ is nonempty, and $\tilde{H}_i(L(\sigma)) = 0$ for all i.

(2). If s is a face of σ, then $L(s) \subset L(\sigma)$.

(3). For each oriented simplex σ, the chains $f_\#(\sigma)$ and $g_\#(\sigma)$ are carried by $L(\sigma)$.

Using these facts, we shall construct the required chain homotopy $D : C_p(K) \to C_{p+1}(L)$, by induction on p. For each σ, the chain $D(\sigma)$ will be carried by $L(\sigma)$.

Let $p = 0$ and v be a vertex of K. Because $f_\#$ and $g\#$ preserve augmentation, $\epsilon(g_\#(v) - f_\#(v)) = 1 - 1 = 0$. Thus $g_\#(v) - f_\#(v)$ represents an element of the reduced homology group $\tilde{H}_0(L(v))$. Because this group vanishes, we can choose a 1-chain $D(v)$ of L carried by the subcomplex $L(v)$ such that

$$\partial(D(v)) = g_\#(v) - f_\#(v).$$

Then $\partial D(v) + D(\partial(v)) = \partial(D(v)) + 0 = g_\#(v) - f_\#(v)$, as desired. Define D in this way for each vertex of K.

Now suppose D is defined in dimensions less than p, such that for each oriented simplex s of dimension less that p, the chain $D(s)$ is carried by $L(s)$, and such that

$$\partial(D(s)) + D(\partial(s)) = g_\#(s) - f_\#(s).$$

Let σ be an oriented simplex of dimension p. We wish to define $D(\sigma)$ so that $\partial(D(\sigma)$ equals the chain

$$c = g_\#(\sigma) - f_\#(\sigma) - D(\partial(\sigma)).$$

Note that c is well-defined chain; $D(\partial(\sigma))$ is defined because $\partial(\sigma)$ has dimension $p - 1$. Furthermore, c is a cycle, for we compute

$$\begin{aligned}
\partial(c) &= \partial(g_\#(\sigma)) - \partial(f_\#(\sigma)) - \partial(D(\partial(\sigma))) \\
&= \partial(g_\#(\sigma)) - \partial(f_\#(\sigma)) \\
&\quad -[g_\#(\partial(\sigma)) - f_\#(\partial(\sigma)) - D(\partial\partial(\sigma)),
\end{aligned}$$

applying the induction hypothesis to the $p - 1$ chain $\partial(\sigma)$. Using the fact that $\partial \circ \partial = 0$, we see that $\partial c = 0$.

Finally we note that c is carried by $L(\sigma)$: Both $g_\#(\sigma)$ and $f_\#(\sigma)$ are carried by $L(\sigma)$, by (3). To show that $D(\partial(\sigma))$ is carried by $L(\sigma)$, note that the chain $\partial(\sigma)$ is a sum of oriented faces of σ. For each such face s, the chain $D(s)$ is carried by $L(s)$, and $L(s) \subset L(\sigma)$ by (2). Thus $D(\partial(\sigma))$ is carried by $L(\sigma)$.

Since c is a p-cycle carried by $L(\sigma)$, and since $H_p(L(\sigma)) = 0$, we can choose a $p + 1$ chain $D(\sigma)$ chain carried by $L(\sigma)$ such that

$$\partial(D(\sigma)) = c = g_\#(\sigma) - f_\#(\sigma) - D(\partial(\sigma)).$$

We then define $D(-\sigma) = -D(\sigma)$. We repeat this process for each p-simplex σ of K; then we have the required chain homotopy D in dimension p. The theorem follows. □

Let K_0 be a subcomplex of K, and L_0 be a subcomplex of L. Let $f : K \to L$ be a simplicial map that carries each simplex of K_0 into a simplex of L_0 then such an f is called a simplicial map of pairs, and we write

$$f : (K, K_0) \to (L, L_0).$$

This map f induces a homomorphism $f_\# : C_*(K) \to C_*(L)$ and its easy to see that $f_\#$ maps $C_*(K_0)$ into $C_p(L_0)$, so we have an induced (also denoted by $f_\#$)

$$f_\# : C_*(K, K_0) \to C_*(L, L_0).$$

Therefore f induces a homomorphism f_* between the relative homology group of K mod K_0 and the relation homology group of L mod L_0

$$f_* : H_*(K, K_0) \to H_*(L, L_0).$$

We prove a theorem analogous to the last theorem about the simplicial maps of pairs.

Definition 2.4 *Let $f, g : (K, K_0) \to (L, L_0)$ be two simplicial maps of pairs. We say f and g are contiguous as maps of pairs if for each simplex $\sigma = v_0 \dots v_p$ of K, the points*

$$f(v_0), \dots, f(v_p), \dots, g(v_0), \dots, g(v_p)$$

span of simplex of L, and if $\sigma \in K_0$, they span a simplex of L_0.

Theorem 2.5 *Let $f, g : (K, K_0) \to (L, L_0)$ be contiguous as maps of pairs. Then there is a homomorphism*

$$D : C_p(K, K_0) \to C_{p+1}(L, L_0)$$

such that $\partial D + D\partial = g_\# - f_\#$. It follows that f_* and g_* are equal as maps of relative homology groups.

Proof: The chain homotopy constructed in the preceding proof automatically maps $C_p(K_0)$ into $C_{p+1}(L_0)$. For if $\sigma \in K_0$), the complex $L(\sigma)$ is by definition a subcomplex of L_0. Given σ, the chain $D(\sigma)$ is carried by $L(\sigma)$; therefore D maps $C_p(K_0)$ into $C_{p+1}(L_0)$. Then D induces the required homomorphism of the relative chain groups. □

Its easy to see that contiguous maps induce same homomorphisms on cohomology groups and relative cohomology groups. We state the following theorem without the proof.

Theorem 2.6 *If $f, g : K \to L$ are contiguous simplicial maps, then*

$$f^* = g^* : H^*(K, G) \to H^*(L, G).$$

3 Barycentric Subdivision

In this section we show that a simplicial complex may be "subdivided" into simplices that are as small as desired and prove that the homology and cohomology groups of the simplicial complex got by subdividing the given complex and the given complex are isomorphic.

Definition 3.1 *A* barycentric subdivision *K' of a simplicial complex K is defined as follows:*

The vertices of K' are the simplices of K, and a finite set of vertices of K' form a simplex of K' if they can be totally ordered by inclusion. That is the simplices of K' are the collection of all simplices of the form

$$\sigma_1\sigma_2...\sigma_n$$

(σ_i is a simplex of K for all i) such that σ_i is a proper face of σ_{i+1} for all i.

Assume that the vertices of K are ordered so that the vertices of any simplex of K have a simple order. Then we define a simplicial map

$$\phi : K' \to K$$

as follows:

If $x' = x_0 x_1 ... x_p$ is a vertex of K', that is, a simplex of K, let $\phi(x')$ be the least vertex of the simplex x' in the given fixed order, i.e., $\phi(x') \in x'$ and, for each $x_i \in x'$, $\phi(x') \leq x_i$. It is easy to see that if we reorder the vertices of K the ϕ will be replaced by a simplicial map contiguous to ϕ. Also, notice that ϕ is order reversing, that is, if $x'_1 \subset x'_2$ then $\phi(x'_1) \geq \phi(x'_2)$.

It is well known that the map ϕ induces an isomorphism (see Eilenberg and Steenrod [2], page 177 - 178) ϕ_* between the homology groups of the simplicial complex and its barycentric subdivision. In fact if (K_1, K_2) is a pair of simplicial complex then (K'_1, K'_2) is again a pair and we can similarly define a map $\phi : (K'_1, K'_2) \to (K_1, K_2)$. This map also induces an isomorphism between the relative homology groups of simplicial complex and its barycentric subdivision.

Let K and L_1 be two simplicial complexes and $\psi : L \to K$ is a simplicial map. Then ψ induces a simplicial map $\psi' : L' \to K'$ of the barycentric subdivisions as follows:

If $y = y_0 ... y_p$ is a vertex of L', that is, a simplex of L, then since ψ is simplicial, the vertices $\psi(y_0) ... \psi(y_p)$ form a simplex of K or a vertex of K'. Let $\psi'(y')$ be this vertex. Clearly ψ' is order preserving, that is, if $y'_1 \subset y'_2$ then $\psi'(y_1) \subset \psi'(y'_2)$, therefore a simplex $y'_0 ... y'_p$ is mapped into a simplex. Thus ψ' is a simplicial map.

Let $\overline{\phi} : L' \to L$ be the map which map each vertex of L' on its least vertex in L.

Lemma 3.2 *The maps $\phi\psi'$ and $\psi\overline{\phi} : L' \to K$ are contiguous.*

Proof: If $y'' = y'_0 ... y'_q$ is a simplex of L', let $\hat{y}'$ be its largest vertex. Then, for each $y'_i \in y''$, $y'_i \subset \hat{y}'$ and hence $\psi' y'_i \subset \psi' \hat{y}'$. Hence, since $\phi\psi'(y'_i) \in \psi'(y'_i)$, $\phi\psi'(y'_i) \in \psi'(\hat{y}')$. Also $\overline{\phi}(y'_i) \in y'_i \subset \hat{y}'$ and hence $\psi\overline{\phi}(y'_i) \in \psi'(\hat{y}')$. Thus the images of the vertices of the simplex y'' both by $\phi\psi'$ and by $\psi\overline{\phi}$ are contained in the simplex $\psi'(\hat{y}')$. □

Chapter 3

Čech and Alexander Cohomology

In this chapter we associate with each topological space X certain graded groups called the Čech cohomology groups and Alexander cohomology groups which are topological invariants of X. We will also define graded groups called Čech homology groups and Vietoris homology groups which also are topological invariants.

1 Direct and inverse limits

We define the notion of *direct limit*:

Definition 1.1 *A* directed set *J is a set with a relation $\leq$ such that*

1. $x \leq x \ \forall\ x \in J$.
2. $x \leq y,\ y \leq x \Rightarrow x = y\ \forall x, y \in J$.
3. $x \leq y,\ y \leq z \Rightarrow x \leq z\ \forall x, y, z \in J$.
4. *for every $x, y \in J$, there exists a $z \in J$ such that $x \leq z$ and $y \leq z$.*

The first three conditions makes J a set with a partial order $\leq$.

Let J be a directed set and $\{A_i \,|\, i \in J\}$ be a collection of additive abelian groups. Suppose for every pair of indices $i, j \in J$ with $i \leq j$ there is a homomorphism $\rho_{ij} : A_i \to A_j$ such that the following hold:

1. $\rho_{jk} \circ \rho_{ij} = \rho_{ik}$ whenever $i \leq j \leq k$, and
2. $\rho_{ii} = 1$ for all $i \in J$.

Let B be a disjoint union of A_i. Define a relation $\sim$ on B by

$$a \sim b \text{ if and only if there exists } k \text{ with } i, j \leq k \text{ and } \rho_{ik}(a) = \rho_{jk}(b),$$

for $a \in A_i$ and $b \in A_j$.

It is easy to see that $\sim$ is an equivalence relation on B. We denote the set of equivalence classes of $\sim$ by $\varinjlim A_i$.

There is naturally a groups structure on $\varinjlim A_i$ with the binary operation defined as follows:

$$\overline{a} + \overline{b} = \overline{\rho_{ik}(a) + \rho_{jk}(b)}.$$

Let $\overline{x}$ denote the class of x in A and define $\rho_i : A_i \to A$ by $\rho_i(a) = \overline{a}$. Clearly if each ρ_{ij} is injective, then so is ρ_i for all i. Thus we may then identify each A_i as a subset of $\varinjlim A_i$.

The group $\varinjlim A_i$ is called the inductive or direct limit of the directed system $\{A_i\}$.

The direct limit of a directed system $\{A_i\}$ has the following universal property:

If C is any abelian group such that for each $i \in J$ there is a homomorphism $\phi : A_i \to C$ with $\phi_i = \phi_j \circ \rho_{ij}$ whenever $i \leq j$, then there is a unique homomorphism $\phi : A \to C$ such that $\phi \circ \rho_i = \phi_i$ for all i.

Observe that if all A_i are commutative rings with identity and all ρ_{ij} are ring homomorphisms that send identity to identity, then $\varinjlim A_i$ may likewise be given the structure of a commutative ring with identity.

We can, in the same way as above, define direct limit for graded groups. All the groups are replaced by graded groups and homomorphism by homomorphisms between graded groups.

We now define the notion of *inverse limit*:

Let I be a partially ordered set (not necessarily directed), and $\{A_i \,|\, i \in J\}$ be a collection of additive abelian groups.

Suppose for every pair of indices $i, j \in I$ with $i \leq j$ there is a map $\mu_{ij} : A_j \to A_i$ such that the following hold:

1. $\mu_{ji} \circ \mu_{kj} = \mu_{ki}$ whenever $i \leq j \leq k$.
2. $\mu_{ii} = 1$ for all $i \in I$.

Let P be a subset of elements $(a_i)_{i\in I}$ in the direct product $\prod_{i\in I} A_i$ such that $\mu_{ji}(a_j) = a_i$ whenever $i \leq j$ (here a_i and a_j are the i^{th} and j^{th} components respectively of the element in the direct product). The set P is denoted by $\varprojlim A_i$. Its easy to see that P is a subgroup of the direct product group $\prod_{i\in I} A_i$. The group $\varprojlim A_i$ is called the projective or inverse limit of the system A_i.

The inverse limit has the following universal property:

If D is any group such that for each $i \in I$ there is a homomorphism $\pi_i : D \to A_i$ with $\pi_i = \mu_{ji} \circ \pi_j$ whenever $i \leq j$, then there is a unique homomorphism $\pi : D \to P$ such that $\mu_i \circ \pi = \pi_i$.

We can, in the same way, define inverse limit for the graded groups.

2 Nerves and Čech Cohomology

Let X be a topological space.

Definition 2.1 *Let $\mathcal{A}$ be a collection of subsets of the space X. We define an abstract simplicial complex called the* nerve *of $\mathcal{A}$, denoted by $N(\mathcal{A})$. Its vertices are the elements of $\mathcal{A}$ and its simplices are the finite subcollections $\{A_1, ..., A_i\}$ of $\mathcal{A}$ such that*

$$A_1 \cap A_2 \cap \ldots \cap A_n \neq \emptyset$$

Its easy to see that $N(\mathcal{A})$ is indeed a simplicial complex.

A collection $\mathcal{B}$ of subsets of X is called a refinement of $\mathcal{A}$ if each element $B \in \mathcal{B}$ is contained in some element $A \in \mathcal{A}$. If $\mathcal{B}$ is a refinement of $\mathcal{A}$, we write $\mathcal{A} < \mathcal{B}$.

Now, let $\mathcal{B}$ be a refinement of $\mathcal{A}$, we can define a map

$$\pi_{\mathcal{BA}} : N(\mathcal{B}) \to N(\mathcal{A})$$

by setting $\pi_{\mathcal{BA}}(B) = A$ such that for $B \in \mathcal{B}$ and $A \in \mathcal{A}$ such that $B \subset A$.

Lemma 2.2 $\pi_{\mathcal{BA}} : N(\mathcal{B}) \to N(\mathcal{A})$ *is a simplicial map.*

Proof: If $\{B_1, ..., B_n\}$ is a simplex of $N(\mathcal{B})$ then

$$\{\pi_{\mathcal{BA}}(B_1), ..., \pi_{\mathcal{BA}}(B_n)\}$$

is a simplex of $N(\mathcal{A})$, because $\cap B_i$ is non empty and is contained in $\pi_{\mathcal{BA}}(B_i)$ for each i. This implies $\cap_i B_i \subset \cap_i \pi_{\mathcal{BA}}(B_i)$ $\square$

Note that the definition of $\pi_{\mathcal{BA}}$ depends on a choice. But we prove that any other choice $\pi'_{\mathcal{BA}}$ is contiguous to $\pi_{\mathcal{BA}}$.

Lemma 2.3 $\pi_{\mathcal{BA}}$ *and* $\pi'_{\mathcal{BA}}$ *are contiguous.*

Proof: We need to show that if $\{B_1, ..., B_n\}$ is a simplex of $N(\mathcal{B})$ then

$$\cap_i \pi_{\mathcal{BA}} \bigcap \cap_i \pi'_{\mathcal{BA}} \neq \emptyset.$$

By definition of $\pi_{\mathcal{BA}}$ and $\pi'_{\mathcal{BA}}$, we have $\cap B_i \subset \pi_{\mathcal{BA}}(B_i)$ for each i and $\cap B_i \subset \pi'_{\mathcal{BA}}(B_i)$ for each i, which gives the desired result. $\square$

Thus we can make the following definition:

Let G be an abelian group.

Definition 2.4 *If $\mathcal{B}$ is a refinement of $\mathcal{A}$, we have uniquely defined homomorphisms,*

$$\pi_{*\mathcal{BA}} : H_*(N(\mathcal{B})) \to H_*(N(\mathcal{A}))$$
$$\pi^*_{\mathcal{BA}} : H^*(N(\mathcal{A}), G) \to H^*(N(\mathcal{B}), G)$$

induced by the simplicial map $\pi_{\mathcal{BA}}$ satisfying the condition $\pi_{\mathcal{BA}}(B) \supset B$ for all $B \in \mathcal{B}$

We now define the groups (graded), Čech cohomology groups and Čech homology groups for a topological space.

Let X be a topological space. Let Ω be a family of open coverings of the space X such that whenever we have, $\mathcal{A}$, $\mathcal{B}$, two coverings of X there is a common refinement $\mathcal{C}$ of $\mathcal{A}$ and $\mathcal{B}$, that is, if there exists $\mathcal{C} \in \Omega$ such that $\mathcal{A} < \mathcal{C}$, $\mathcal{B} < \mathcal{C}$. Then the collection of graded groups $\{H^*(N(\mathcal{A}), G) \mid \mathcal{A} \in \Omega\}$ along with the homomorphisms

$$\pi^*_{\mathcal{B}\mathcal{A}} : H^*(N(\mathcal{A}), G) \to H^*(N(\mathcal{B}), G)$$

for coverings $\mathcal{B}$ and $\mathcal{A}$ with $\mathcal{A} < \mathcal{B}$, forms a directed system.

Similarly the collection of graded groups $\{H_*(N(\mathcal{A}) \mid \mathcal{A} \in \Omega\}$ along with the homomorphisms

$$\pi_{*\mathcal{B}\mathcal{A}} : H_*(N(\mathcal{B})) \to H_*(N(\mathcal{A}))$$

for coverings $\mathcal{B}$ and $\mathcal{A}$ with $\mathcal{A} < \mathcal{B}$, forms an inverse system.

Definition 2.5 *(a). The* Čech cohomology group*, $\check{H}^*(X, G)$, of X is the graded group*

$$\varinjlim_{\mathcal{A}\in\Omega} H^*(N(\mathcal{A}), G)$$

(b). The Čech homology group*, $\check{H}_*(X)$, of X is the graded group*

$$\varprojlim_{\mathcal{A}\in\Omega} H_*(N(\mathcal{A}).$$

3 Vietoris complex and Alexander Cohomology

Let X be a topological space.

Definition 3.1 *Let $\mathcal{A}$ be a collection of subsets of the space X. We define an abstract simplicial complex called the* Vietoris complex *denoted by $V(\mathcal{A})$. The vertices of $V(\mathcal{A})$ are the points of X and the simplices of $V(\mathcal{A})$ are finite set of elements of X contained in a common element of $\mathcal{A}$.*

It is easy to see that $V(\mathcal{A})$ is indeed a simplicial complex.

Now, let $\mathcal{B}$ be a refinement of $\mathcal{A}$. Define a map $\overline{\pi}_{\mathcal{B}\mathcal{A}} : V(\mathcal{B}) \to V(\mathcal{A})$ by setting $\overline{\pi}_{\mathcal{B}\mathcal{A}}(x) = x$ for $x \in X$.

It is easy to see that this map is a simplicial map and by the very definition of the map it is clear that it induces a unique homomorphism on the homology and cohomology groups of the simplicial complexes.

Definition 3.2 *If $\mathcal{B}$ is a refinement of $\mathcal{A}$, we have uniquely defined homomorphisms,*

$$\overline{\pi}_{*\mathcal{B}\mathcal{A}} : H_*(V(\mathcal{B})) \to H_*(V(\mathcal{A}))$$
$$\overline{\pi}^*_{\mathcal{B}\mathcal{A}} : H^*(N(\mathcal{A}), G) \to H^*(N(\mathcal{B}), G)$$

induced by the simplicial map $\overline{\pi}_{\mathcal{B}\mathcal{A}}$.

We now define the graded groups: *Alexander cohomology group* and *Vietoris homology group* for a topological space.

Let X be a topological space. Let Ω be a family of open coverings of the space X such that whenever we have, $\mathcal{A}$, $\mathcal{B}$, two coverings of X there is a common refinement $\mathcal{C}$ of $\mathcal{A}$ and $\mathcal{B}$. Then the collection of graded groups $\{H^*(V(\mathcal{A}), G) \,|\, \mathcal{A} \in \Omega\}$ along with the homomorphisms

$$\overline{\pi}^*_{\mathcal{B}\mathcal{A}} : H^*(V(\mathcal{A}), G) \to H^*(V(\mathcal{B}), G)$$

for coverings $\mathcal{B}$ and $\mathcal{A}$ with $\mathcal{A} < \mathcal{B}$, forms a directed system.

Similarly the collection of graded groups $\{H_*(V(\mathcal{A}) \,|\, \mathcal{A} \in \Omega\}$ along with the homomorphisms

$$\overline{\pi}_{*\mathcal{B}\mathcal{A}} : H_*(V(\mathcal{B})) \to H_*(V(\mathcal{A}))$$

for coverings $\mathcal{B}$ and $\mathcal{A}$ with $\mathcal{A} < \mathcal{B}$, forms an inverse system.

Definition 3.3 *(a). The* Alexander cohomology group*, $\tilde{H}^*(X, G)$, of X is the graded group*

$$\varinjlim_{\mathcal{A}\in\Omega} H^*(V(\mathcal{A}), G)$$

(b). The Vietoris homology group*, $\tilde{H}_*(X)$, of X is the graded group*

$$\varprojlim_{\mathcal{A}\in\Omega} H_*(V(\mathcal{A}).$$

Chapter 4

Homology of Relations

In the previous chapter we defined Čech and Alexander cohomology groups of a topological space, we also defined Čech and Vietoris homology groups. In this chapter we prove that the two homology groups and two cohomology groups are isomorphic. We in fact prove a more general theorem, due to C. H. Dowker [1]. The isomorphisms of the homology and cohomology groups mentioned above are proved as a consequence of the Dowker's Theorem.

1 Complexes associated with relations

Let X and Y be sets. A relation R is a subset of the cartesian product $X \times Y$. If $(x, y) \in R$ we write $x\,R\,y$ and say that x is related to y. Consider a relation R, we define two simplicial complexes K and L associated to the relation R as follows:

A finite subset s of X is a simplex of K if there exists a $y_s \in Y$ such that $x\,R\,y_s$ for all $x \in s$.

A finite subset t of Y is a simplex of L if there exists a $x_t \in X$ such that $x_t\,R\,y$ for all $y \in t$.

Lemma 1.1 *K and L are simplicial complexes.*

Proof: We need to show that if s is a simplex of K then every subset of s is also a simplex of K. Suppose s is a simplex of K and s' a subset of s; there exists $y \in Y$ such that $x\,R\,y$ for all $x \in s$, which implies that $x\,R\,y$ for all $x \in s'$ since $s' \subset s$. The vertex set V of K_X is the set of all those points of X which are related to atleast one element of Y. Therefore K is a simplicial complex. Similarly, L is a simplicial complex. □

Definition 1.2 *Let X_1, Y_1 be sets with relation R_1 and X_2, Y_2 be sets with relation R_2. A map $f : (X_2, Y_2) \to (X_1, Y_1)$, that is a map f mapping X_2 into X_1, Y_2 into Y_1 is called a map of relation R_2 into the relation R_1 if whenever $x \in X_2$ and $y \in Y_2$ with $x\,R_2\,y$, we have $f(x)\,R_1\,f(y)$.*

Let X_1, Y_1 be sets with relation R_1 and X_2, Y_2 be sets with relation R_2 and $f : (X_2, Y_2) \to (X_1, Y_1)$ be a map of relation R_2 into the relation R_1. Let K_1, L_1 and K_2, L_2 be the simplicial complexes corresponding to the relations R_1 and R_2 respectively.

If s is a simplex of K_2; there exists $y_s \in Y_2$ such that $x\,R_2\,y_s$ for all $x \in s$. Then by definition of f, we have $f(x)\,R_1\,f(y_s)$ for all $x \in s$. This implies that $\{f(x)|x \in s\}$ form a simplex of K_1. Now define

$$f_{21} : K_2 \to K_1$$

by $f_{21}(x) = f(x)$ for a vertex x of K. Then by the above argument f_{21} is easily seen to be a simplicial map. We call f_{21} the simplicial map induced by f. Similarly f induces another simplicial map

$$\overline{f}_{21} : L_2 \to L_1.$$

Let X_3, Y_3 be sets with a relation R_3 and $g : (X_3, Y_3) \to (X_2, Y_2)$ be a map of relation R_3 into R_2, then we can compose the two map f and g to get a map $f \circ g : (X_3, Y_3) \to (X_1, Y_1)$ which is a map of relation R_3 into the relation R_1.

Lemma 1.3 $(f \circ g)_{31} = f_{21} \circ g_{32}$ *and* $\overline{f \circ g}_{31} = \overline{f}_{21} \circ \overline{g}_{32}$.

Proof: For $x \in X_3$,

$$(f \circ g)_{31}(x) = f \circ g(x) = f(g(x)) = f(g_{32}(x)) = f_{21}(g_{32}(x)) = f_{21} \circ g_{31}(x)$$

similarly, $\overline{(f \circ g)}_{31} = \overline{f}_{21} \circ \overline{g}_{32}$. □

If X_2 is a subset of X_1, Y_2 is a subset of Y_1 and R_2 is a subset of $R_1 \cap (X_2 \times Y_2)$. Then R_2 is called a subrelation of R_1. (R_1, R_2) is called a pair of relations. The inclusion map $i : (X_2, Y_2) \hookrightarrow (X_1, Y_1)$ induces simplicial maps $i_{21} : K_2 \to K_1$ and $\overline{i}_{21} : L_2 \to L_1$. Note that K_2 is a subcomplex of K_1 and L_2 is a subcomplex of L_1 in this case. Thus (K_1, K_2) and (L_2, L_1) are pairs associated to the pair (R_1, R_2).

Let X_{α_1}, Y_{α_1} be sets with subsets X_{α_2}, Y_{α_2} and $(R_{\alpha_1}, R_{\alpha_2})$ be the pair of relations between the sets. Let $(R_{\beta_1}, R_{\beta_2})$ be another pair of relations between the sets X_{β_1} and Y_{β_1} with subsets X_{β_2} and $Y_{\beta 2}$. Let K_{α_1}, K_{α_2} and L_{α_1}, L_{α_2} be the simplicial complexes associated with the pair $(R_{\alpha_1}, R_{\alpha_2})$ and Let K_{β_1}, K_{β_2} and L_{β_1}, L_{β_2} be the simplicial complexes associated to the pair $(R_{\beta_1}, R_{\beta_2})$

Definition 1.4 *A map $f : (X_{\alpha_1}, X_{\alpha_2}, Y_{\alpha_1}, Y_{\alpha_2}) \to (X_{\beta_1}, X_{\beta_2}, Y_{\beta_1}, Y_{\beta_2})$ is called a map of pair if f maps X_{α_1}, X_{α_2}, Y_{α_1}, Y_{α_2} into X_{β_1}, X_{β_2}, Y_{β_1}, Y_{β_2} respectively and whenever $x \, R_{\alpha_1} \, y$ we have $f(x) \, R_{\beta_1} \, f(y)$.*

Observe that, if f is map of pair as above then the submap

$$f : (X_{\alpha_1}, Y_{\alpha_1}) \to (X_{\beta_1}, Y_{\beta_1})$$

induces a simplicial map $f_{\alpha\beta_1} : K_{\alpha_1} \to K_{\beta_1}$ and the submap

$$f : (X_{\alpha_2}, Y_{\alpha_2}) \to (X_{\beta_2}, Y_{\beta_2})$$

induces a simplicial map $f_{\alpha\beta_2} : K_{\alpha_2} \to K_{\beta_2}$.

If $i_\alpha : K_{\alpha_2} \hookrightarrow K_{\alpha_1}$, $i_\beta : K_{\beta_2} \hookrightarrow K_{\beta_1}$ are inclusion maps then it is easy to see that

$$i_\beta f_{\alpha\beta_2} = f_{\alpha\beta_1} i_\alpha : K_{\alpha_2} \to K_{\beta_1}.$$

Thus f induces a simplicial map

$$f_{\alpha\beta} : (K_{\alpha_1}, K_{\alpha_2}) \to (K_{\beta_1}, K_{\beta_2})$$

of the pair of simplicial complexes $(K_{\alpha_1}, K_{\alpha_2})$, $(K_{\beta_1}, K_{\beta_2})$.

Similarly f induces a simplicial map

$$\overline{f}_{\alpha\beta} : (L_{\alpha_1}, L_{\alpha_2}) \to (L_{\beta_1}, L_{\beta_2})$$

of the pair of simplicial complexes $(K_{\alpha_1}, K_{\alpha_2})$, $(K_{\beta_1}, K_{\beta_2})$.

2 The homomorphisms η and ω

We first recall that a barycentric subdivision K' of a simplicial complex K is defined as follows:

The vertices of K' are the simplices of K and a finite set of vertices of K' form a simplex of K' if they can be simply ordered by inclusion.

Also recall that the simplicial map $\phi : K' \to K$ is defined as follows:

First order the vertices of K such that the vertices of any simplex of K have a total order. Then, if $x' = x_0 \ldots x_p$ is a vertex of K', that is a simplex of K, define $\phi(x')$ to be equal to the least vertex of x'.

We know that ϕ induces an isomorphism on the homology and cohomology groups of K' and K. That is

$$\phi_* : H_*(K') \to H_*(K) \text{ and } \phi^* : H^*(K) \to H^*(K')$$

are isomorphisms.

Now, Let R be a relation between the sets X and Y and let K, L be the simplicial complexes associated to the relation R as defined in section 1. Let $\phi : K' \to K$ and $\overline{\phi} : L' \to L$ be the simplicial maps defined as above.

Define a map $\psi : L' \to K$ from the first barycentric subdivision L' of L to K as follows:

Let y' be a vertex of L', then by definition of barycentric subdivision y' is a simplex of L; choose $x \in X$ such that $x\,R\,y$ for all $y \in y'$ (by definition of L such a choice is possible). Define $\psi(y') = x$.

Lemma 2.1 $\psi : L' \to K$ *defined above is a simplicial map.*

Proof: We need to show that if $y'' = y'_0 ... y'_q$ is a simplex of L', then $\{\psi(y'_0), ..., \psi(y'_q)\}$ form a simplex of K, that is we need to show that there exists some $y \in Y$ such that $\psi(y'_i)\, R\, y$ for all i. Let $\tilde{y}'$ be the least vertex of y''. Since $\overline{\phi}(\tilde{y}')$ is the least vertex of $\tilde{y}'$, for each $y'_i \in y''$, $\overline{\phi}(\tilde{y}') \in \tilde{y}' \subset y'_i$. This implies that $\psi(y'_i)\, R\, \overline{\phi}(\tilde{y}')$ which completes the proof. $\square$

Note that the definition of ψ depends on the choice of the element $x \in X$ as above. We show that if $\hat{\psi}$ is the result of some other choice $\hat{x} \in X$, then ψ and $\hat{\psi}$ are contiguous.

Lemma 2.2 *ψ and $\hat{\psi}$ are contiguous.*

Proof: We have, $\hat{\psi}(y'_i)\, R\, \overline{\phi}(\tilde{y}')$ for each i. Thus $\psi(y'_0), ..., \psi(y'_q), \hat{\psi}(y'_0), ..., \hat{\psi}(y'_q)$ are the vertices of a simplex of K. This implies that ψ and $\hat{\psi}$ are contiguous. $\square$

By the above lemmas we conclude that ψ induces homomorphisms

$$\psi_* : H_*(L') \to H_*(K) \text{ and } \psi^* : H^*(K) \to H^*(L')$$

on the homology and cohomology groups of L' and K and that this homomorphism is uniquely determined.

We know that

$$\overline{\phi}_* : H_*(L') \to H_*(L) \text{ and } \overline{\phi}^* : H^*(L) \to H^*(L')$$

are isomorphisms. Consider

$$\overline{\phi}_*^{-1} : H_*(L) \to H_*(L') \text{ and } \overline{\phi}^{*-1} : H^*(L') \to H^*(L)$$

and define

$$\eta = \overline{\phi}^{*-1} \circ \psi^* : H^*(K) \to H^*(L)$$

and

$$\omega = \psi_* \circ \overline{\phi}_*^{-1} : H_*(L) \to H_*(K).$$

Note that if (R_1, R_2) is a pair of relations and (K_1, K_2) and (L_1, L_2) are the pairs of simplicial complexes associated with the pair of relations. Then we can do the same as above with these pairs too. That is:

We have a simplicial map $\psi : (L'_1, L'_2) \to (K_1, K_2)$ and it induces homomorphisms

$$\psi_* : H_*(L'_1, L'_2) \to H_*(K_1, K_2) \text{ and } \psi^* : H^*(K_1, K_2) \to H^*(L'_1, L'_2)$$

which are uniquely determined.

We have isomorphisms

$$\overline{\phi}_* : H_*(L'_1, L'_2) \to H_*(L_1, L_2) \text{ and } \overline{\phi}^* : H^*(L_1, L_2) \to H^*(L'_1, L'_2)$$

and we define

$$\eta = \overline{\phi}^{*-1} \circ \psi^* : H^*(K_1, K_2) \to H^*(L_1, L_2)$$

and

$$\omega = \psi_* \circ \overline{\phi}_*^{\ -1} : H_*(L_1, L_2) \to H_*(K_1, K_2).$$

We have the following two lemmas:

Let $\eta_2 = (\overline{\phi}|_{L'_2})^{*^{-1}} \circ (\psi|_{L'_2})^* : H^*(K_2) \to H^*(L_2)$

Lemma 2.3 *The homomorphism η commutes with the coboundary homomorphism, that is*

$$\eta\delta = \overline{\delta}\eta_2 : H^p(K_2) \to H^{p+1}(L_1, L_2).$$

Proof: By the third axiom of cohomology theory of Eilenberg and Steenrod (see Appendix B), which says that the coboundary homomorphism is a natural transformation, we have the following diagram which is commutative in each rectangle:

$$\begin{array}{ccccc}
H^p(K_2) & \xrightarrow{(\psi|_{L'_2})^*} & H^p(L'_2) & \xrightarrow{(\overline{\phi}|_{L'_2})^{*^{-1}}} & H^p(L_2) \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \overline{\delta}'} & & \downarrow{\scriptstyle \overline{\delta}} \\
H^{p+1}(K_1, K_2) & \xrightarrow{\psi^*} & H^{p+1}(L'_1, L'_2) & \xrightarrow{\overline{\phi}^*} & H^{p+1}(L_1, L_2)
\end{array}$$

.

Hence

$$\overline{\phi}^{*-1}\psi^*\delta = \overline{\delta}(\overline{\phi}|_{L'_2})^{*^{-1}}(\psi|_{L'_2})^*$$

that is, $\eta\delta = \overline{\delta}\eta_2$, as desired. $\square$

Let $\omega_2 = (\psi|_{L_2'})_* \circ {(\overline{\phi}|_{L_2'})_*}^{-1} : H_p(L_2) \to H_p(K_2)$

Lemma 2.4 *The homomorphism ω commutes with the boundary homomorphism, that is,*

$$\partial\omega = \omega_2\overline{\delta} : H_p(L_1, L_2) \to H_{p-1}(K_2).$$

Proof: The proof is similar to the proof of last lemma. By the third axiom of homology theory of Eilenberg and Steenrod (see Appendix B), the boundary homomorphism is a natural transformation, the boundary homomorphism commutes with the homomorphisms ψ_* and $\overline{\phi}_*$, which gives the desired result. $\square$

Lemma 2.5 *If f is a map of relations R_β into R_α then $f_{\beta\alpha}\psi_\beta$ and $\psi_\alpha\overline{f}'_{\beta\alpha} : L_\beta' \to K_\alpha$ are contiguous.*

Proof: Let $y'' = y_0'...y_q'$ be a simplex of L_β', and let $\tilde{y}'$ be its least vertex. Let y_0 be an element of $\tilde{y}'$. Then, for each $y_i' \in y''$, $y_0 \in \tilde{y}' \subset y_i'$, and hence $\psi_\beta(y_i')\, R_\beta\, y_0$. Therefore $f(\psi_\beta(y_i'))\, R_\alpha\, f(y_0)$, that is, $f_{\beta\alpha}\psi_\beta(y_i')\, R_\alpha\, y_0$. Also, since $y_0 \in y_i'$, $f(y_0) = \overline{f}(y_0) \in \overline{f}'_{\beta\alpha}(y_i')$ and hence $\psi_\alpha\overline{f}'_{\beta\alpha}(y_i')\, R_\alpha\, f(y_0)$. Hence all the vertices $f_{\beta\alpha}\psi_\beta(y_i')$ and $\psi_\alpha\overline{f}'_{\beta\alpha}(y_i')$ belong to a common simplex of K_α. Therefore $f_{\beta\alpha}\psi_\beta$ and $\psi_\alpha\overline{f}'_{\beta\alpha}$ are contiguous. $\square$

Lemma 2.6 *If f is a map of relations R_β into R_α then f^* commutes with η, that is*

$$\eta_\beta f^*_{\beta\alpha} = \overline{f}^*_{\beta\alpha}\eta_\alpha : H^*(K_\alpha) \to H^*(L_\beta).$$

Proof: We have the following diagram.

$$\begin{array}{ccccc}
H^p(K_\alpha) & \xrightarrow{\psi_\alpha^*} & H^p(L_\alpha') & \xrightarrow{{\overline{\phi}_\alpha^*}^{-1}} & H^p(L_\alpha) \\
\downarrow{\scriptstyle f^*_{\beta\alpha}} & & \downarrow{\scriptstyle \overline{f}'^*_{\beta\alpha}} & & \downarrow{\scriptstyle \overline{f}^*_{\beta\alpha}} \\
H^p(K_\beta) & \xrightarrow{\psi_\beta^*} & H^p(L_\beta') & \xrightarrow{{\overline{\phi}_\alpha^*}^{-1}} & H^p(L_\beta)
\end{array}$$

It follows from the last lemma that commutativity in the left rectangle holds. and from the lemma (insert lemma number) it follows that the commutativity holds in the right rectangle. Hence $\phi_\beta^{*^{-1}}\psi_\beta^* f_{\beta\alpha}^* = \overline{f}_{\beta\alpha}^* \overline{\phi}_\alpha^{*^{-1}} \psi_\alpha^*$. Which gives $\eta_\beta f_{\beta\alpha}^* \eta_\alpha$. ☐

Lemma 2.7 *If f is a map of relations R_β into R_α then f^* commutes with ω, that is*

$$f_{\beta\alpha_*}\omega_\beta = \omega_\alpha \overline{f}_{\beta\alpha_*} : H^*(L_\beta) \to H^*(K_\alpha).$$

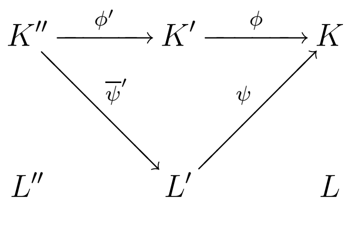
Proof: The proof is similar to the proof of last lemma. We omit the proof of this lemma. ☐

3 A Theorem of Dowker

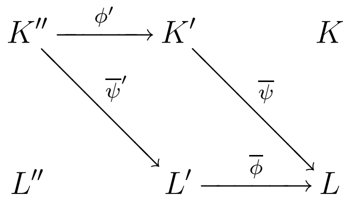
In this section we prove the general theorem, due to Dowker [1], that the homomorphisms η and ω defined in the previous section are in fact isomorphisms. That is, we show that the two simplicial complexes associated to a relation have isomorphic homology and cohomology groups.

Let R be a relation between the sets X and Y and K, L be the two simplicial complexes associated with R. Let K'' and L'' be second barycentric subdivisions of K and L respectively. We have the following simplicial maps $\phi : K' \to K$, $\phi' : K'' \to K'$, $\overline{\phi} : L' \to L$, $\overline{\phi} : L'' \to L$, $\psi : L' \to K$, $\overline{\psi}' : K'' \to L'$, $\overline{\psi} : K' \to L$ defined as in the last section.

We have the following diagrams:



and



Lemma 3.1 *The simplicial maps $\psi \circ \overline{\psi}'$ and $\phi \circ \phi' : K'' \to K$ are contiguous.*

Proof: Let $x''' = x_0''...x_q''$ be a simplex of K''. We need to show that there exists a $y \in Y$ such that $\phi\phi(x_i'')\, R\, y$ and $\psi\psi'(x_i'')\, R\, y$.

Let $\tilde{x}''$ be the least vertex of x'''. For each x_i'', $\tilde{x}'' \subset x_i''$. Since ϕ' is order reversing, we have $\phi'(x_i'') \subset \phi'(\tilde{x}'')$. Let $y = \overline{\psi}\phi'(\tilde{x}'')$; then $x\, R\, y$ for each $x \in \phi' x_i''$. Hence, since $\phi\phi'(x_i'') \in \phi'(x_i'') \subset \phi'(\tilde{x}'')$, $\phi\phi'(x_i'')\, R\, y$.

For each vertex x' of x_i'', $\overline{\psi}(x') \in \overline{\psi}'(x_i'')$. Hence, since $\phi'(\tilde{x}'') \in \tilde{x}'' \subset x_i''$, $\overline{\psi}\phi'(\tilde{x}'') \in \overline{\psi}'(x_i'')$. Thus $y \in \overline{\psi}'(x_i'')$ for each x_i''. Hence, for each x_i'', $\psi\overline{\psi}'(x_i'')\, R\, y$, as desired. □

Lemma 3.2 *The simplicial maps $\overline{\phi}\,\overline{\psi}'$ and $\overline{\psi}\phi' : K'' \to L$ are contiguous.*

Proof: The proof is immediate from lemma (3.2) of chapter 2. □

We now prove the main theorem of this section:

Theorem 3.3 *If R is a relation between the sets X and Y and K and L are the simplicial complexes associated with the relation R, then*

$$\eta : H^*(K) \to H^*(L)$$

is an isomorphism.

Proof: By the previous two lemmas we have the following results:

$$(\psi\overline{\psi}')^* = (\phi\phi')^* : H^*(K) \to H^*(K'')$$

That is

$$\overline{\psi}'^{*}\psi^{*} = \phi'^{*}\phi^{*} : H^{*}(K) \to H^{*}(K'').$$

But since $\psi^{*}\psi'^{*}$ are isomorphisms,$\overline{\psi}'^{*}\psi^{*}$ is an isomorphism. Thus we have

$$\phi^{*-1}\phi'^{*-1}\overline{\psi}'^{*}\psi^{*} = 1 : H^{*}(K) \to H^{*}(K). \tag{4.1}$$

We also have

$$\overline{\psi}'^{*}\overline{\phi}^{*} = \phi'^{*}\overline{\psi}^{*} : H^{*}(L) \to H^{*}(K'')$$

which gives

$$\phi'^{*-1}\overline{\psi}'^{*} = \overline{\psi}^{*}\overline{\phi}^{*-1} : H^{*}(L') \to H^{*}(K')$$

Substituting this in (4.1) we get,

$$\phi^{*-1}\overline{\psi}^{*}\overline{\phi}^{*-1}\psi^{*} = 1.$$

But since $\eta = \overline{\phi}^{*-1}\psi^{*}$, we have

$$\overline{\eta}\eta = 1 : H^{*}(K) \to H^{*}(K)$$

where $\overline{\eta} = \phi^{*-1}\overline{\psi}^{*}$.

Similarly we get,

$$\eta\overline{\eta} = 1 : H^{*}(L) \to H^{*}(L)$$

which shows that η is an isomorphism. □

Theorem 3.4 *If R is a relation between the sets X and Y and K and L are the simplicial complexes associated with the relation R, then*

$$\omega : H_{*}(L) \to H_{*}(K)$$

is an isomorphism.

Proof: The proof is similar to that of the previous theorem. We omit the proof of this theorem. □

4 An Application of Dowker's Theorem

We give an application of Dowker's theorem. We prove that the Alexander cohomology groups and the Čech cohomology groups are isomorphic for arbitrary topological spaces. We also prove that the Čech homology group and Vietoris homology group are also isomorphic for arbitrary topological spaces.

Let X be a topological space and Ω be a family of open coverings of X such that whenever we have two coverings $\mathcal{A}$,$\mathcal{B} \in \Omega$ there is a common refinement $\mathcal{C}$ of $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{A} \in \Omega$. Treating $\mathcal{A}$ as a set we define a relation R between X and $\mathcal{A}$ as follows:

for $x \in X$ and $U \in \mathcal{A}$, $x\,R\,U$ if and only if $x \in U$, that is if x is an element of U.

Lemma 4.1 *The complexes K and L associated to the relation R are Vietoris complex and nerve of the covering $\mathcal{A}$ respectively.*

Proof: A simplex of K is the finite set of points of X which is related to some element U of $\mathcal{A}$. That is, if the finite set of point belongs to U. But this is precisely the definition of a simplex of the Vietoris complex.

A simplex of L is the finite set of elements $\{U_1, ..., U_n\}$ of $\mathcal{A}$ which is related to some element of X. That is, if there exist some $x \in X$ such that $x \in U_i$ for all i. That is, if $\cap U_i \neq \emptyset$. But this is precisely the definition of a simplex of nerve of $\mathcal{A}$. □

Let $\mathcal{B} \in \Omega$ such that $\mathcal{A} < \mathcal{B}$, that is $\mathcal{B}$ is a refinement of $\mathcal{A}$.

Let $R_{\mathcal{A}}$, $R_{\mathcal{B}}$ be the relations defined as above between X and $\mathcal{A}$ and X and $\mathcal{B}$ respectively. Let $K_{\mathcal{A}}$, $L_{\mathcal{A}}$ and $K_{\mathcal{B}}$, $L_{\mathcal{B}}$ be the simplicial complexes associated to the relations $R_{\mathcal{A}}$ and $R_{\mathcal{B}}$ respectively.

Define a relation map $\pi : (X, \mathcal{B}) \to (X, \mathcal{A})$ by setting $\pi(x) = x$ for $x \in X$ and $\pi(U) = V$ for $U \in \mathcal{A}$ and $V \in \mathcal{B}$ with $V \subset U$. π induces simplicial maps $\overline{\pi}_{\mathcal{B}\mathcal{A}} : K_{\mathcal{B}} \to K_{\mathcal{A}}$ and $\pi_{\mathcal{B}\mathcal{A}} : L_{\mathcal{B}} \to L_{\mathcal{A}}$.

Since K and L are the Vietoris complex and nerve of the coverings respectively the simplicial maps defined above are same as the simplicial maps defined in the section (2) and (3) of chapter 3.

Let G be an abelian group.

We know that $\pi_{\mathcal{BA}}$ and $\overline{\pi}_{\mathcal{BA}}$ induce homomorphisms

$$\overline{\pi}^*_{\mathcal{BA}} : H^*(K_{\mathcal{A}}, G) \to H^*(K_{\mathcal{B}}, G),$$
$$\overline{\pi}_{*\mathcal{BA}} : H_*(K_{\mathcal{B}}) \to H_*(K_{\mathcal{A}}),$$
$$\pi^*_{\mathcal{BA}} : H^*(L_{\mathcal{A}}, G) \to H^*(L_{\mathcal{A}}, G)$$

and

$$\pi_{*\mathcal{BA}} : H_*(L_{\mathcal{B}}) \to H_*(L_{\mathcal{A}}).$$

We know that all these homomorphisms are uniquely determined.

The groups $\{H^*(K_{\mathcal{A}}, G) \,|\, \mathcal{A} \in \Omega\}$, $\{H^*(L_{\mathcal{A}}, G) \,|\, \mathcal{A} \in \Omega\}$ with the homomorphisms $\overline{\pi}^*$, π^* respectively, form directed systems. And we know that the limit groups $\varinjlim_{\mathcal{A}\in\Omega} H^*(K_{\mathcal{A}}, G)$ and $\varinjlim_{\mathcal{A}\in\Omega} H^*(L_{\mathcal{A}}, G)$ are the Alexander cohomology group and Čech cohomology group of X respectively.

Theorem 4.2 *The Čech and Alexander cohomology groups are isomorphic.*

Proof: By theorem (3.3) of the last section $\eta_1 : H^*(K_{\mathcal{A}}, G) \to H^*(L_{\mathcal{A}}, G)$ and $\eta_2 : H^*(K_{\mathcal{B}}, G) \to H^*(L_{\mathcal{B}}, G)$ are isomorphisms. By lemma (2.6) π^* commutes with η, we have

$$\eta_2 \overline{\pi}^*_{\mathcal{BA}} = \pi^*_{\mathcal{BA}} \eta_1 : H^*(K_{\mathcal{B}}, G) \to H^*(L_{\mathcal{A}}, G).$$

Thus, identifying $H^*(K_{\mathcal{A}}, G)$ and $H^*(L_{\mathcal{A}}, G)$, the two directed systems $\{H^*(K_{\mathcal{A}}, G) \,|\, \mathcal{A} \in \Omega\}$ and $\{H^*(L_{\mathcal{A}}, G) \,|\, \mathcal{A} \in \Omega\}$ are identified. And therefore the limit groups are also identified. □

The groups $\{H_*(K_{\mathcal{A}}) \,|\, \mathcal{A} \in \Omega\}$, $\{H_*(L_{\mathcal{A}}) \,|\, \mathcal{A} \in \Omega\}$ with the homomorphisms $\overline{\pi}_*$, π_* respectively, form inverse systems. And we know that the limit groups $\varprojlim_{\mathcal{A}\in\Omega} H_*(K_{\mathcal{A}})$ and $\varprojlim_{\mathcal{A}\in\Omega} H_*(L_{\mathcal{A}}, G)$ are the Vietoris homology group and Čech homology group of X respectively.

Theorem 4.3 *The Čech and Vietoris homology groups are isomorphic.*

Proof: The proof is similar to the proof of last theorem and so we omit the proof. □

Appendix A

Categories, Functors and Natural Transformations

Definition 1 *A category* **C** *consists of a class of* objects *and sets of* morphisms *between those objects. For every ordered pair A and B of objects there is a set $Hom(A, B)$ of morphisms from A to B, and for every ordered triple A, B, C of objects there is a law of composition of morphisms, i.e., a map*

$$Hom(A, B) \times Hom(B, C) \to Hom(A, C)$$

where $(f, g) \mapsto gf$, gf is called the composition of g with f. The objects and morphism satisfy the following axioms: for objects A, B, C and D

1. if $A \neq B$ or $C \neq D$, then $Hom(A, B)$ and $Hom(C, D)$ are disjoint sets.

2. composition of morphism is associative, i.e., $h(gh) = (hg)f$ for every $f \in Hom(A, B)$, $g \in Hom(B, C)$ and $h \in Hom(C, D)$,

3. each object has an identity morphism, i.e., for every object A there is a morphism $1_A \in Hom(A, A)$ such that $f1_A = f$ for every $f \in Hom(A, B)$ and $1_A g = g$ for every $g \in Hom(B, A)$.

Examples.

1. **Grp** is the category of all groups, where morphism are groups homomorphisms.

2. **Top** is the category of all topological spaces and morphism are continuous between topological spaces.

Definition 2 *Let* **C** *and* **D** *be categories.*

1. *We say $\mathcal{F}$ is a covariant functor from* **C** *to* **D** *if*

(a) *for every object A in* **C**, *$\mathcal{F}A$ is an object in* **D**, *and*

(b) *for every $f \in Hom(A, B)$ we have $\mathcal{F}(f) \in Hom(\mathcal{F}A, \mathcal{F}B)$ such that*

(i). *if gf is a composition of morphism in* **C**, *then $\mathcal{F}(gf) = \mathcal{F}(g)\mathcal{F}(f)$ in* **D** *and*

(ii). $\mathcal{F}(1_A) = 1_{\mathcal{F}A}$

2. *We say $\mathcal{F}$ is a contravariant functor from* **C** *to* **D** *if the conditions in (1) hold but property (b) and axiom (i) are replaced by:*

(b'). *for every $f \in Hom(A, B)$, $\mathcal{F}(f) \in Hom(\mathcal{F}B, \mathcal{F}A)$,*

(i'). *if gf is a composition of morphisms in* **C***, then $\mathcal{F}(gf) = \mathcal{F}(f)\mathcal{F}(g)$ in* **D***.*

Definition 3 *Let* **C** *and* **D** *be categories and let $\mathcal{F}$, $\mathcal{G}$ be covariant functors from* **C** *to* **D***. A* natural transformation *from $\mathcal{F}$ and $\mathcal{G}$ is a map η that assigns to each object A in* **C** *a morphism η_A in $Hom(\mathcal{F}A, \mathcal{G}A)$ with the following property: for every pair of objects A and B in* **C** *and every $f \in Hom(A, B)$ we have $\mathcal{G}(f)\eta_A = \eta_B\mathcal{F}(f)$, i.e., the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{F}A & \xrightarrow{\eta_A} & \mathcal{G}A \\ {\scriptstyle \mathcal{F}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{G}(f)} \\ \mathcal{F}B & \xrightarrow{\eta_B} & \mathcal{G}B \end{array}$$

Appendix B

The Eilenberg-Steenrod Axioms

Historically, the homology groups of simplicial complexes were the first homology groups defined and studied. Later, various generalized definitions of homology were formulated but they all gave same results as simplicial homology. This led Eilenberg and Steenrod to axiomatize the notion of a homology theory.

Definition 4 *Let $\mathcal{A}$ be a class of pairs (X, A) of topological spaces such that:*

1. *If (X, A) belongs to $\mathcal{A}$, so do (X, X), $(X, \emptyset)$, (A, A) and $(A, \emptyset)$.*
2. *If (X, A) belongs to $\mathcal{A}$, so does $(X \times I, A \times I)$. (I closed interval.)*
3. *There is a one-point space P such that $(P, \emptyset)$ is in $\mathcal{A}$.*

Then we call A an admissible class of spaces for a homology theory.

Definition 5 *If $\mathcal{A}$ is admissible, a* homology theory *on A consists of three function:*

1. A function H_p defined for each integer p and each pair (X, A) in $\mathcal{A}$, whose value is an abelian group.

2. A function that, for each integer p, assigns to each continuous map $h : (X, A) \to (Y, B)$ a homomorphism

$$(h_*)_p : H_p(X, A) \to H_p(Y, B).$$

3. A function that, for each integer p, assigns to each pair (X, A) in $\mathcal{A}$, a homomorphism

$$\partial_p : H_p(X, A) \to H_{p-1}(A),$$

where A denotes the pair $(A, \emptyset)$.

There functions are to satisfy the following axioms, where all pairs of space are in $\mathcal{A}$. For convenience we delete the dimensional subscripts on h_* and ∂.

Axiom 1. *If i is the identity map, then i_* is the identity homomorphism.*

Axiom 2. $(k \circ h)_* = k_* \circ h_*$.

Axiom 3. *If $f : (X, A) \to (Y, B)$, then the following diagram commutes*

$$\begin{array}{ccc} H_p(X,A) & \xrightarrow{f_*} & H_p(Y,B) \\ \partial \downarrow & & \downarrow \partial \\ H_{p-}(A) & \xrightarrow{(f|_A)_*} & H_{p-1}(B) \end{array}$$

Axiom 4 *(Exactness axiom). The sequence*

$$\cdots \ H_p(A) \xrightarrow{i_*} H_p(X) \xrightarrow{\pi_*} H_p(X, A) \xrightarrow{\partial} H_{p-1}(A) \cdots$$

, where $i : A \hookrightarrow X$ and $\pi : X \hookrightarrow (X, A)$ are inclusion maps, is exact, i.e., the composition of any two adjacent homomorphisms is a zero homomorphism and kernel of the homomorphism on right side is equal to the image of the homomorphism on left.

Given two maps $h, k : (X, A) \to (Y, B)$ are said to be homotopic if there is a map $F : (X \times I, A \times I) \to (Y, B)$ such that $F(x, 0) = h(x)$ and $F(x, 1) = k(x)$ for all $x \in X$.

Axiom 5 *(Homotopy axiom). If h and k are homotopic, then $h_* = k_*$.*

Axiom 6 *(Excision Axiom). Given (X, A), let U be an open subset of X such that $\overline{U} \subset Int A$. If $(X - U, A - U)$ is admissible, then inclusion induces isomorphism*

$$H_p(X - U, A - U) \cong H_p(X, A).$$

Axiom 7 *(Dimension Axiom). If P is a one-point space, then $H_p(P) = 0$ for $p \neq 0$ and $H_0(P) = \mathbb{Z}$.*

The functions H_p are in fact covariant functors from the category of admissible pairs of topological spaces to the category of abelian groups. We can also say that $H_* = \{H_p\}$ is a covariant functor from the category of admissible pairs of topological spaces to the category of graded groups. It is easily seen from the third axiom that ∂ is a natural transformation from the functor H_p to the functor H_{p-1}.

Eilenberg and Steenrod axiomatized the cohomology theory also in the similar fasion.

It is known that the simplicial homology theory satisfies all the axioms stated above.

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