Bayesian games, Repeated Games, Information Design and Persuasion

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Outline

Bayesian games

A Bayesian game, or game with incomplete information is given by:

- A finite set of players $i = 1, \dots, N$;
- A set of states $\Omega = \Theta \times \prod_i M_i$ with a prior probability distribution $p \in \Delta(\Omega)$;
- A set of actions A_i for each player i;
- A payoff function $u_i : \Omega \times A \to \mathbb{R}$ for each player i.

1. A state $\omega = (\theta, m_1, \dots, m_N)$ is drawn from p, Player i is informed of m_i.
2. Players choose actions simultaneously.

The game at the second stage is possibly dynamic: actions represent pure strategies in the continuation (extensive form) game.

Bayes-Nash equilibrium

A Bayes-Nash equilibrium is a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_N)$ with $\sigma_i : M_i \to \Delta(A_i)$ such that:

For each player i, each message m_i with $p(m_i) > 0$, each action a_i,

$$\mathbb{E}[u_i(\theta, m_i, m_{-i}, \sigma_i(m_i), \sigma_{-i}(m_{-i})) \mid m_i] \geq \mathbb{E}[u_i(\theta, m_i, m_{-i}, a_i, \sigma_{-i}(m_{-i})) \mid m_i]$$

This is an Nash equilibrium of the normal form game with payoff $\mathbb{E}[u_i(\theta, m, \sigma(m))]$.

This exists when:

- States and actions are finite;

- States are finite or countable, actions sets are compact metric, payoffs are continuous.

Outline

Zero-sum games

We consider two-player zero-sum games with independent types (for simplicity).

- There are two players, the finite state space is $K \times L$ with prior distribution $p(k)q(l)$.
- Action sets are A, B. Payoffs are $u_1 = -u_2 = g(k, l, a, b)$.

The game has a value if

$$\sup_\sigma \inf_\tau \mathbb{E}_{p,q}[g(k, l, \sigma, \tau)] = \inf_\tau \sup_\sigma \mathbb{E}_{p,q}[g(k, l, \sigma, \tau)] := V(p, q)$$

- If $|L| = 1$ or $|K| = 1$, this is a game with Lack of Information on one-side. (Otherwise, on both sides).

Lack on one-side $|L| = 1$

Examples: $K = \{k_0, k_1\}$, $p = (\frac{1}{2}, \frac{1}{2})$. Let G_k be the payoff matrix in state k.

$$G_{k_0} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad G_{k_1} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

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$$G_{k_0} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad G_{k_1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$G_{k_0} = \begin{pmatrix} 4 & 0 & 2 \\ 4 & 0 & -2 \end{pmatrix} \qquad G_{k_1} = \begin{pmatrix} 0 & 4 & -2 \\ 0 & 4 & 2 \end{pmatrix}$$

Concavity

Let $G(p)$ be the game with prior distribution p and let $V(p)$ be its value.

Theorem (Aumann Maschler)

$V(\cdot)$ is a concave function of p.

This follows from the following statement:

Suppose that for each q, player 1 guarantees $f(q)$ in $G(q)$ and let $p = \sum_m \lambda_m p_m$. Then, player 1 guarantees $\sum_m \lambda_m f(p_m)$ in $G(p)$.

The theorem follows with $f(q) = V(q)$:

$$V(p) \geq \sum_m \lambda_m V(p_m)$$

We prove now this statement.

Experiments and splittings

Let M be a finite set of messages. A statistical (Blackwell) experiment is $x : K \to \Delta(M)$. We have

$$\mathbb{P}(k, m) = p(k)x(m|k) = \lambda_m p_m(k)$$

where $\lambda_m = \mathbb{P}(m) = \sum_k p(k)x(m|k)$ and

$$p_m(k) = \mathbb{P}(k \mid m) = \frac{\mathbb{P}(k, m)}{\mathbb{P}(m)} = p(k)x(m|k)/\lambda_m$$

Also,

$$p(k) = \sum_m \mathbb{P}(k, m) = \sum_m \lambda_m p_m(k)$$

Splitting

A splitting of $p \in \Delta(K)$ is a distribution (random posterior) $\tilde{p}$ such that $\mathbb{E}(\tilde{p}) = p$. With finite support, this is convex combination

$$p = \sum_m \lambda_m p_m$$

Any experiment induces a splitting.

Splitting lemma

Conversely, since

$$\mathbb{P}(k, m) = p(k)x(m|k) = \lambda_m p_m(k)$$

Splitting lemma (Aumann Maschler, 65)

Any splitting of p

$$p = \sum_m \lambda_m p_m$$

is induced by the experiment $x(m|k) = \lambda_m p_m(k)/p(k)$

Auxiliary game

- Consider the game $\tilde{G}(p)$ where player 1 can send a message m to player 2 before choosing actions. Let $\tilde{V}(p)$ be its value.
- If $p = \sum_m \lambda_m p_m$, then player 1 can send the message m and guarantee $f(p_m)$ afterwards. So $\tilde{V}(p) \geq \sum_m \lambda_m f(p_m)$.
- Consider now the game $\tilde{G}^*(p)$ where player 2 does not observe the message. Player 2 has less strategies in $\tilde{G}^*(p)$ than in $\tilde{G}(p)$. So, $\tilde{V}^*(p) \geq \tilde{V}(p)$.
- But then $\tilde{G}^*(p) \equiv G(p)$. So

$$V(p) \geq \tilde{V}(p) \geq \sum_m \lambda_m f(p_m)$$

QED.

Concavification

Let $u(p)$ be the value of the game without information:

$u(p)$ is the value of the matrix game $D(p) = \sum_k p(k) G_k$.

Let $\operatorname{Cav} u(p)$ be its concave closure:

$$\begin{aligned} \operatorname{Cav} u(p) &= \inf\{f(p) : f \geq u, f \text{ concave}\} \\ &= \sup\left\{ \sum_m \lambda_m u(p_m) : \sum_m \lambda_m p_m = p \right\} \end{aligned}$$

Lemma

$$V(p) \geq \operatorname{Cav} u(p)$$

Repeated game

$G_n(p)$ is the n-stage repeated game with average payoff where:

- State k is drawn from p once and for all and known by player 1;
- Actions are chosen at each stage and are observed; Payoffs are not observed.

$V_n(p)$ is the value of $G_n(p)$. $V_\infty(p)$ is the (uniform) value of $G_\infty(p)$ (if it exists).

Theorem (Aumann Maschler)

1. $\text{Cav}\, u(p) \leq V_n(p) \leq \text{Cav}\, u(p) + \frac{cte}{\sqrt{n}}$
2. $V_\infty(p) = \text{Cav}\, u(p)$.

We know that $V_n(p), V_\infty(p) \geq \text{Cav}\, u(p)$.

Examples

$$G_{k_0} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad G_{k_1} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

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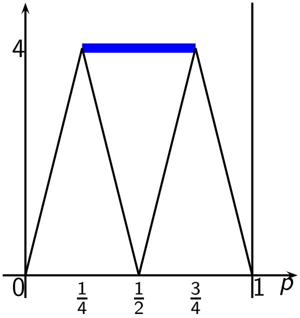
$u(p) = \max\{p, 1-p\}$ convex. Full revelation.

$$G_{k_0} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad G_{k_1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$u(p) = p(1-p)$ concave. No revelation.

Examples

$$G_{k_0} = \begin{pmatrix} 4 & 0 & 2 \\ 4 & 0 & -2 \end{pmatrix} \qquad G_{k_1} = \begin{pmatrix} 0 & 4 & -2 \\ 0 & 4 & 2 \end{pmatrix}$$



Partial revelation

$V_n(p) \leq \mathrm{Cav}\, u(p) + \frac{cte}{\sqrt{n}}$

- Fix a strategy σ of player 1 and a history of actions $h_n = (a_1, b_1, \ldots, a_{n-1}, b_{n-1})$. Let $p_n(k) = \mathbb{P}(k|h_n)$.
- Interpret $\sigma(a_n = m|k, h_n)$ as an experiment $x(m|k)$.
- $\lambda_m = \sum_k p_n(k)x(m|k)$ is the probability of playing $a_n = m$.
- Let $\tau(h_n)$ be a best-response to λ in $D(p_n)$:

$$\sum_k p_n(k) G_k(\lambda, \tau(h_n)) \leq u(p_n)$$

We have

$$\begin{aligned}
\mathbb{E}[G_k(a_n, b_n)|h_n] &\leq \sum_k p_n(k) G_k(\lambda, \tau(h_n)) + C \sum_k p_n(k) \sum_m |x(m|k) - \lambda_m| \\
&\leq u(p_n) + C \sum_m \lambda_m \sum_k |p_{n+1}(k) - p_n(k)|
\end{aligned}$$

Then (with the Martingale property),

$$\mathbb{E}[\bar{G}_n] \leq \mathbb{E}\frac{1}{n}\sum_t u(p_t) + \frac{cte(p)}{\sqrt{n}} \leq \mathrm{Cav}\, u(p) + \frac{cte(p)}{\sqrt{n}}$$

$V_\infty(p) \leq \operatorname{Cav} u(p)$

We need to prove that player 2 guarantees $\operatorname{Cav} u(p)$ in the infinite (limit) game.

- There exists a vector $\xi \in \mathbb{R}^K$ such that

$$\operatorname{Cav} u(p) = \langle p, \xi \rangle \text{ and } \forall q, u(q) \leq \langle q, \xi \rangle$$

Given history h_n, define a vector $v \in \mathbb{R}^k$ with $v_k = (1/n) \sum_t G_k(a_t, b_t)$. Define $q \in \Delta(K)$ by

$$q_k = \frac{(v_k - \xi_k)_+}{\sum_j (v_j - \xi_j)_+}$$

- If q is well defined, player 2 plays an optimal strategy in $D(q)$.
- Player 2 plays arbitrarily otherwise.

$V_\infty(p) \leq \operatorname{Cav} u(p)$

Let $v_n = (1/n)\sum_t G(a_t, b_t)$ and $\pi(v_n)$ be the projection on $C = \{w \leq \xi\}$.

By construction, $v_n - \pi(v_n) = \lambda q$ with $\lambda > 0$ and

$$\langle \mathbb{E}[G(a_{n+1}, b_{n+1}) \mid h_n], q\rangle \leq u(q) \leq \langle q, \xi\rangle = \langle q, \pi(v_n)\rangle$$

For any strategy σ of player 1, if $v_n \notin C$,

$$\mathbb{E}[\langle G(a_{n+1}, b_{n+1}) - \pi(v_n), v_n - \pi(v_n)\rangle \mid h_n] \leq 0$$

If a sequence of vectors is such that

$$\langle x_{n+1} - \pi(\bar{x}_n), \bar{x}_n - \pi(\bar{x}_n)\rangle \leq 0$$

then $d(\bar{x}_n, C) \to 0$. QED

Outline

Sender Receiver Games

A Sender Receiver game is a two-player Bayesian game where:

- The set of states is Θ with a prior probability p.
- Player 1 knows the state and chooses a message $m \in M$.
- Player 2 does not know the state and observes m. Then he chooses an action $a \in A$.
- Player 1's payoff is $u(\theta, a)$, player 2's payoff is $v(\theta, a)$.

Player 1 is an expert, Player 2 is a decision maker.

Information is transmitted strategically.

Since utilities are different, information is transmitted imperfectly.

Example

Consider the Sender-Receiver game defined by:

	a_1	a_2	
θ_1	$(1,5)$	$(0,1)$	$\frac{1}{2}$
θ_2	$(1,-10)$	$(0,1)$	$\frac{1}{2}$

- Player 1 only cares about player 2 taking action a_1.
- For player 2, a_2 is a safe bet, a_1 is a risky bet.
- The only equilibrium is babbling (non-informative).

Equilibrium condition for player 2

- A behavior strategy for player 1 is $\sigma : \Theta \to \Delta(M)$ denoted $\sigma(m \mid \theta)$.
- A strategy σ of player 1 is formally an experiment and is equivalent to a splitting $p = \sum_m \lambda_m p_m$.
- A behavior strategy for player 2 is $\tau : M \to \Delta(A)$ denoted $\tau(a \mid m)$.
- Player 2 who receives the message m has belief p_m.

Player 2 with belief p over the states, chooses a mixed action from the set:

$$Y(p) = \left\{ \alpha \in \Delta(A) : \sum_{\theta} p(\theta) v(\theta, \alpha) = \max_a \sum_{\theta} p(\theta) v(\theta, a) \right\}$$

- At a best-reply, player 2 chooses $y_m \in Y(p_m)$.

Equilibrium condition for player 1

Suppose that player 2 chooses $y_m \in \Delta(A)$ after receiving message m.

Consider the best-reply of player 1.

- For each state θ and each message m,

$$\sigma(m \mid \theta) > 0 \Rightarrow u(\theta, y_m) = \max_{m'} u(\theta, y_{m'}) := U_\theta$$

- Notice that $\sigma(m \mid \theta) > 0 \Leftrightarrow p_m(\theta) > 0$. Player 1 is indifferent between all messages (and thus posteriors) induced with positive probability.

- So for each m, $u(\theta, y_m) \leq \max_{m'} u(\theta, y_{m'}) = U_\theta$ with equality if $p_m(\theta) > 0$.

Equilibrium characterization

Consider the set $\mathcal{E}$ of tuples $(p, U, V) \in \Delta(\Theta) \times \mathbb{R}^{\Theta} \times \mathbb{R}$, such that $\exists y \in \Delta(A)$:

(i) $U_\theta \geq u(\theta, y)$ with equality if $p(\theta) > 0$,

(ii) $y \in Y(p)$,

(iii) $V = \sum_\theta p(\theta) v(\theta, y)$.

Denote

$$\operatorname{conv}_U(\mathcal{E}) = \left\{ \sum \lambda_m (p_m, U, V_m) : \forall m, (p_m, U, V_m) \in \mathcal{E} \right\}$$

Theorem (Aumann-Hart, Forges)

There is an equilibrium of $G(p)$ with payoff (U, V) if and only if $(p, U, V) \in \operatorname{conv}_U(\mathcal{E})$

Compare with $\operatorname{Cav} u(p)$. Convexify the non-revealing payoffs. Indifference condition for player 1.

Full revelation

A game with full revelation in equilibrium.

	a_1	a_2
θ_1	$(1,1)$	$(0,0)$
θ_2	$(0,0)$	$(3,3)$

No revelation

A game with no revelation in equilibrium.

	a_1	a_2	a_3	
θ_1	$(3, -4)$	$(2, -1)$	$(1, 0)$	p
θ_2	$(3, 0)$	$(2, -1)$	$(1, -4)$	$1 - p$

- We have,

$$Y(p) = \begin{cases} \{a_1\} \text{ if } p < 1/4 \\ \{a_2\} \text{ if } 1/4 < p < 3/4 \\ \{a_3\} \text{ if } p > 3/4 \end{cases}$$

- It is not possible to split p and keep player 1 indifferent.

Partial revelation

A game with partial revelation in equilibrium.

	a_1	a_2	a_3	a_4	a_5	
θ_1	$(1, 10)$	$(3, 8)$	$(0, 5)$	$(3, 0)$	$(1, -8)$	p
θ_2	$(1, -8)$	$(3, 0)$	$(0, 5)$	$(3, 8)$	$(1, 10)$	$1 - p$

We have,

$$Y(p) = \begin{cases} \{a_1\} \text{ if } p > 4/5 \\ \{a_2\} \text{ if } 4/5 > p > 5/8 \\ \{a_3\} \text{ if } 5/8 > p > 3/8 \\ \{a_4\} \text{ if } 3/8 > p > 1/5 \\ \{a_5\} \text{ if } 1/5 > p \end{cases}$$

Partial revelation

- For $p = 1/4$, player 2 plays a_4 and player 1's payoff are $(3, 3)$.
- For $p = 3/4$, player 2 plays a_2 and player 1's payoff are $(3, 3)$.
- Player 1 can induce those beliefs by the splitting

$$\frac{1}{2} = (1/2)\frac{1}{4} + (1/2)\frac{3}{4}$$

- Or by the strategy $\sigma(m_1 \mid \theta_1) = \frac{3}{4} = \sigma(m_2 \mid \theta_2)$.
- This equilibrium payoff dominates NR and CR.
- Player 1 obtains the maximal payoff 3.

Outline

Bayesian persuasion

Consider a sender-receiver environment $p \in \Delta(\Theta)$, $u(\theta, a)$, $v(\theta, a)$.

- Player 1 is an *information designer* if he can choose the information structure $x : \Theta \to \Delta(M)$ which delivers a message to player 2.
- Player 1 is no longer informed of the state, but chooses how information is disclosed to player 2.
- The sender receiver game is played à la Stackelberg:
  - Player 1 chooses the experiment x once and for all. It is publicly observed.
  - Nature draws θ from p and m from x.
  - Player observes x and chooses an action.
- Player 1 is able to commit on its randomizations: distributions of messages are observable and verifiable.

This model is called Bayesian Persuasion (Kamenica and Gentzkow, 2011).

Receiver

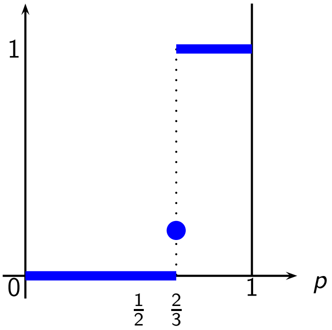
Player 2 with belief p over the states, chooses a mixed action from the set:

$$Y(p) = \left\{ \alpha \in \Delta(A) : \sum_{\theta} p(\theta) v(\theta, \alpha) = \max_{a} \sum_{\theta} p(\theta) v(\theta, a) \right\}$$

An equilibrium strategy of player 2 is a tie-breaking-rule (TBR): a selection $\gamma(p) \in Y(p)$.

Example

	a_1	a_2	
θ_1	$(1,5)$	$(0,1)$	$\frac{1}{2}$
θ_2	$(1,-10)$	$(0,1)$	$\frac{1}{2}$



Solving the game

A strategy σ of player 1 is an experiment and is equivalent to a splitting $p = \sum_m \lambda_m p_m$.

Given that the receiver chooses TBR $\gamma(p)$, the program of the sender is

$$\max \left\{ \sum_m \lambda_m \sum_\theta p_m(\theta) u(\theta, \gamma(p_m)) : p = \sum_m \lambda_m p_m \right\}$$

Denote $U_\gamma(p) = \sum_\theta p(\theta) u(\theta, \gamma(p))$ then,

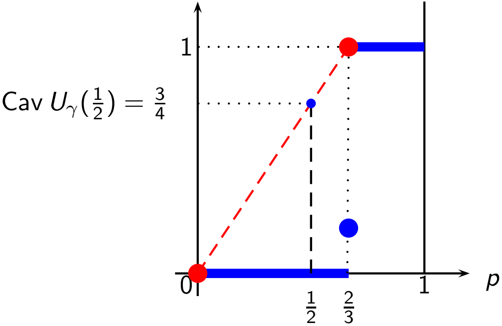
Lemma (Kamenica-Gentzkow, 11)

The equilibrium payoff of player 1 is

$$\sup \left\{ \sum_m \lambda_m U_\gamma(p_m) : p = \sum_m \lambda_m p_m \right\} = \text{Cav}\, U_\gamma(p)$$

Example

	a_1	a_2	
θ_1	$(1,5)$	$(0,1)$	$\frac{1}{2}$
θ_2	$(1,-10)$	$(0,1)$	$\frac{1}{2}$



Outline

Games between information designers

Environment

- Finite multidimensional state space $\Theta = \Theta_1 \times \cdots \times \Theta_n$.
- Prior $p^0 \in \Delta(\Theta)$ (possibly correlated).
- Designer $i = 1, \ldots, n$ chooses a statistical experiment $x_i : \Theta_i \to \Delta(M_i)$ (*information structure*), where $M_i = M_i^1 \times \cdots \times M_i^k$ is a finite set of message profiles.
- Each agent $j = 1, \ldots, k$ chooses an action $a_j \in A_j$.
- Designer i has payoff $u_i(a, \theta_1, \ldots, \theta_n)$, with $a \in A = A_1 \times \cdots \times A_k$.
- Agent j has payoff $v_j(a, \theta_1, \ldots, \theta_n)$.

Timing of the $(n+k)$-player Information Design Game G_M

Finite sets of messages are given.

1. Designers choose experiments $(x_1, \ldots, x_n)$ simultaneously.
2. Agents publicly observe $(x_1, \ldots, x_n)$.

A k-player Bayesian subgame $G_M(x)$ is then played:

3. Nature draws the state $\theta = (\theta_1, \ldots, \theta_n)$ according to $p^0 \in \Delta(\Theta)$, and a uniformly distributed sunspot $\omega \in [0, 1]$.
4. A profile of messages $m_i = (m_i^1, \ldots, m_i^k)$ from each designer i is drawn with probability $x_i(m_i \mid \theta_i)$.
5. Agents publicly observe ω, each agent j privately observes the profile of messages $m^j \in M^j = \prod_{i \in N} M_i^j$ and chooses an action a_j.

Subgame Perfect Equilibrium

For each profile of experiments of the designers $x \in \prod_i \Delta(M_i)^{\Theta_i}$, the Bayesian game $G_M(x)$ is a finite game, so its set of Nash equilibrium outcomes

$$\mathcal{E}_M(x) \subseteq \{y : M \to \Delta(A)\} = \Delta(A)^M$$

is non-empty.

Since there is public correlation, $\mathcal{E}_M(x)$ is convex (set of public correlated equilibria of $G_M(x)$).

Theorem (Existence, Koessler et al. 2021)

For every profile of finite message sets, the $(n+k)-$player information design G_M game admits a subgame perfect equilibrium $(x, y), y \in \mathcal{E}_M(x)$.

Relies on Simon and Zame (1990).

Outline

Splitting games

- There are two finite sets K, L and initial "states" $p^0 \in P = \Delta(K)$, $q^0 \in Q = \Delta(L)$. At every stage n, in states (p^n, q^n),
- Player 1 chooses a splitting $s \in S(p^n) = \left\{ s \in \Delta(P) : E_s(\boldsymbol{p}^n) = p^n \right\}$,
- Player 2 chooses a splitting $t \in S(q^n) = \left\{ t \in \Delta(Q) : E_t(\boldsymbol{q}^n) = q^n \right\}$,
- the next states (p^{n+1}, q^{n+1}) are drawn from $s \otimes t$.
- The stage payoff is $u(p^n, q^n)$ with $u : P \times Q \to \mathbb{R}$ continuous.

Discounted game: $\sum \lambda(1-\lambda)^{n-1} u(p^n, q^n)$.

Long game: $u(p^\infty, q^\infty)$.

Splitting games

- Splitting games are games of information design: Player 1 designs information about k, Player 2 about l.

  At the end of the day, a decision maker takes an optimal decision $z(p, q)$ as a function of his posterior beliefs (p, q).

  $u(p, q)$ is the payoff of Player 1 induced by this decision.
- These are special stochastic games with "acyclic" transitions.

Back to repeated games with incomplete information

- Splitting games were introduced as proxy for repeated games with incomplete information on both sides where:

  - Player 1 knows k, Player 2 knows l,

  - they repeatedly choose actions i, j which are observed,

  - payoffs G_{ij}^{kl} are unobserved.

  - Let,

$$u(p, q) = \mathrm{Val}\Big(\sum_{k,l} p^k q^l G_{ij}^{kl} \Big)$$

Concave–Convex functions

What is the correct generalization of $\operatorname{Cav} u$?

- $\operatorname{Cav}_p \operatorname{Vex}_q u(p,q)$. This is the MaxMin of the undiscounted repeated game with incomplete information on both sides.
- $\operatorname{Vex}_q \operatorname{Cav}_p u(p,q)$. This is the MinMax of the undiscounted repeated game with incomplete information on both sides.
- In general

  $$\operatorname{Cav}_p \operatorname{Vex}_q u(p,q) < \operatorname{Vex}_q \operatorname{Cav}_p u(p,q)$$

  Therefore the undiscounted repeated game has no value.

Mertens-Zamir

Yet,

Theorem (Mertens-Zamir, 1971)

In the repeated game with incomplete information on both sides, $v(p,q) := \lim v_\lambda(p,q)$ exists and is the unique continuous function such that

$$v(p,q) = \text{Cav}\,\min(u,v)(p,q) = \text{Vex}\,\max(u,v)(p,q)$$

- v is concave–convex and $\text{Cav}_p\text{Vex}_q u(p,q) < v(p,q) < \text{Vex}_q\text{Cav}_p u(p,q)$.
- Sketch of proof: As long as $u(p,q) \geq v(p,q)$, Player 1 stays silent.

  As soon as $u(p,q) < v(p,q)$, Player 1 Plays optimally in the discounted game.

  Player 1 thus gets a convex combination of $\max(u,v)(p_t,q_t)$ which is $\geq \text{Vex}\,\max(u,v)(p,q)$.
- MZ prove existence and uniqueness of v.

Splitting game

In the splitting game with stage payoff $u(p, q)$, then

Theorem

$v = \lim v_\lambda$ (Laraki 2001). It is also the value of the undiscounted game (Oliu Barton 2017) and of the long game (Koessler et al. 2021).

One can show that v is the unique function such that

- If $u(p, q) < v(p, q)$, there exists $v \in S(p)$ such that $v(p, q) = v(s, q) \geq u(s, q)$
- If $u(p, q) > v(p, q)$, there exists $t \in S(q)$ such that $v(p, q) = v(p, t) \leq u(p, t)$

This defines an optimal strategy for each player.

Approximation of the MZ function

Assume $|K| = |L| = 1$.

- Consider grids $P^n = \{0, \frac{1}{n}, \dots, \frac{i}{n}, 1\}$, $Q^m = \{0, \frac{1}{m}, \dots, \frac{j}{m}, 1\}$ on $[0,1] = \Delta(K) = \Delta(L)$.
- Let $U = (u_{ij})$ be the matrix $u_{i,j} = u(\frac{i}{n}, \frac{j}{m})$.

Lemma (Koessler et al. 2021)

There exists a unique matrix V which is column-concave, row-convex such that $v_{00} = u_{00}$, $v_{01} = u_{01}$, $v_{10} = u_{10}$, $v_{11} = u_{11}$ and,

$$\text{if } v_{i,j} > u_{i,j}, \text{ then } v_{i,j} = \frac{1}{2}(v_{i-1,j} + v_{i+1,j}),$$

$$\text{if } v_{i,j} < u_{i,j}, \text{ then } v_{i,j} = \frac{1}{2}(v_{i,j-1} + v_{i,j+1}).$$

- This matrix can be found by solving linear systems.
- This induces a piecewise linear function which uniformly approximates the function u.

Thank you !

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