Externalities in One-Sided Markets

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Joint work with Sagar Massand

Allocation problem

Allocation of indivisible items

- A finite set of agents N.
- A finite set of indivisible items A.
- Question. Does there exist a "good" allocation of items to agents?

Desirable allocation criteria

- Fairness:
  - envy-free allocation,
  - proportional allocation,
  - max-min share guarantee.
- Stability.

One-sided markets

House allocation problem [Shapley and Scarf]

- A set of agents N and a set of items A.
- Assume that $|N| = |A|$.
- Preference ordering over the items $(\prec_i)_{i \in N}$.
- An allocation $\pi : N \to A$ (an injective function).

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Related models

- Stable matching problem (Gale and Shapley) - two types of agents.
- Roommates problem - there are no types.

Stability

Stable outcomes

- 2-stable: An allocation where no pair of players is able to improve their individual utilities by mutual exchange of items.
- core-stable: An allocation where no coalition of players is able to improve the utilities of all the players involved by exchanging items.

Top-trading cycle algorithm

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- Let π^* be an arbitrary initial allocation.
- **Step 1.** Given an allocation π, create a graph on the item set as follows:
  - $x \longrightarrow y$, if y is the most-preferred item for agent $\pi^{-1}(x)$ from the items in the graph at the moment.

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  - $x \rightarrow y$, if y is the most-preferred item for agent $\pi^{-1}(x)$ from the items in the graph at the moment.
- **Step 2.** For any cycle $(x_1, x_2, \ldots, x_k, x_1)$, assign x_i to $\pi^{-1}(x_{i-1})$ (taking $x_k = x_{1-1}$). Remove these items.
- Repeat steps 1 & 2 until there are no items remaining in the graph.

Graphical one-sided markets

- Agents utilities depend on:
  - valuation for the allocated item,
  - items which are allocated to other agents (within a neighbourhood).
- Agents have cardinal utilities.
- Utilities are non-transferable.

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$$u_i(\pi) = v_i(\pi(i)) + r_i(\pi)$$

Requirement: A compact representation for r_i.

Graphical one-sided markets

The model - Graphical Matching Problem (GMP)

- A set of agents N and a set of items A where $|A| = |N|$.
- A directed agent graph $G = (N, \tau, w)$,
  - $w_{i,j}$ denotes the weight of edge (i, j).
- An undirected item graph $H = (A, \lambda)$.

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  - $w_{i,j}$ denotes the weight of edge (i, j).
- An undirected item graph $H = (A, \lambda)$.
- $i_\tau = \{j \mid (j, i) \in \tau\}$ - neighbourhood of agent i.
- $a_\lambda = \{b \mid (a, b) \in \lambda\}$ - neighbourhood of item a.

Graphical one-sided markets

Allocation

- Allocation - a bijection $\pi : N \rightarrow A$.
- $N(i, \pi) = \{j \in N \mid \pi(j) \in (\pi(i))_\lambda\}$ - agent *i*'s neighbours under allocation π.
- $d_i(\pi) = i_\tau \cap N(i, \pi)$ - agents influencing *i*'s utility.

Utility

- $r_i(\pi) = \sum_{j \in d_i(\pi)} w_{j,i}$ - externalities on a player *i*.
- $v_i : A \rightarrow R_{\geq 0}$ - intrinsic valuation over items for agent *i*.
- $u_i(\pi) = v_i(\pi(i)) + r_i(\pi)$ - agent *i*'s final utility.

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Uniform valuation - $v_i(a) = c$ for all $i \in N$ and $a \in A$.

Example

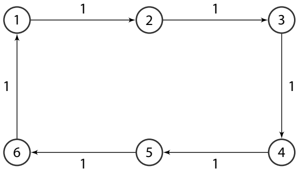
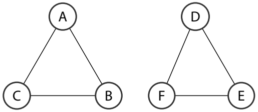


Figure: Agent graph



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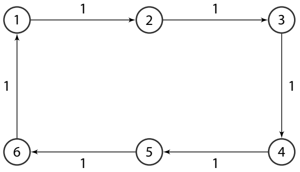

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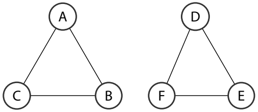

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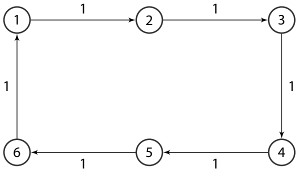
Suppose for all $i \in N$, for all $a \in A$, $v_i(a) = 0$.

- An allocation $\pi = ((1, 2, 3)), (4, 5, 6))$.
- $N(1, \pi) = \{3, 2\}$, $d_1(\pi) = \emptyset$, $u_1(\pi) = 0$.

Example



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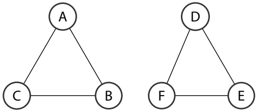

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- $N(1, \pi) = \{3, 2\}, d_1(\pi) = \emptyset, u_1(\pi) = 0$.
- $N(2, \pi) = \{1, 3\}, d_2(\pi) = \{1\}, u_2(\pi) = 1$.

Related frameworks

Related models

- Bouveret *et al.*, 2017: Fair division of a graph.
- Chevaleyre *et al.*, 2017: Distributed fair allocation of indivisible goods.
- Ghodsi *et al.*, 2018: Fair allocation of indivisible items with externalities.
- Elkind *et al.*, 2019: Schelling games on graphs.

Stability

Blocking pair

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2-stable allocation

An allocation π is 2-stable if it has no blocking pair.

No pair of players is able to improve their individual utilities by mutual exchange of items.

Stability

Blocking coalition

An allocation π has a blocking coalition $X \subseteq N$ ($X \neq \emptyset$) if there exists a bijection $\mu : X \to X$ such that $\forall i \in X, \pi'(i) = \pi(\mu(i))$ with $u_i(\pi') > u_i(\pi)$.

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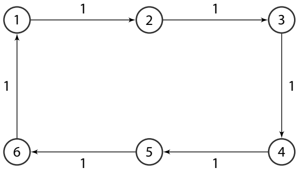
- No neighbourhood externalities: A core stable outcome always exists and can be computed in polynomial time [Shapley, Scarf].
- With neighbourhood externalities: A 2-stable allocation need not always exists.

Example



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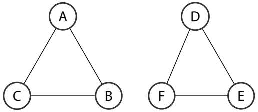

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- $((1, 2, 3)), (4, 5, 6))$.
- $((1, 2, 4)), (3, 5, 6))$.
- $((1, 2, 5), (3, 4, 6))$.

Stable allocation

Theorem. For a *symmetric* instance of GMP, a 2-stable allocation always exists.

Proof idea

- Solution - local search.
- Proof of termination.
  - $\varphi(\pi) = \sum_{i \in N} u_i(\pi) + v_i(\pi)$ is a progress measure.

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Corollary. In a symmetric GMP with uniform valuation, where the underlying player graph is unweighted, we can compute a 2-stable allocation in polynomial time.

Proof idea

- $2nc \leq \varphi(\pi) \leq 2nc + 2|\tau|$.
- Each resolution step increases the progress measure by at least two.

Computation of 2-stable allocation

The general case

Theorem. Computation of 2-stable allocations is PLS-complete even for symmetric instances of GMP with uniform valuation.

Theorem. Local search takes exponential time in worst case to compute a 2-stable allocation.

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Degree two graphs

For a symmetric instance of GMP with uniform valuation with the degree of the agent graph being bounded by two, there exists a polynomial time algorithm to compute a 2-stable allocation.

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Proof idea

Polynomially bounded potential function expressed in terms of the degree of each agent. $\varphi(\pi) = \sum_{i \in N}(2n - g(i))f_i(u_i(\pi))$ where $f_i(u_i(\pi)) \in \{0, 1, 2, 3\}$.

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Bounded degree graphs

Computation of 2-stable allocations is PLS-complete even for symmetric instances of GMP with uniform valuation even when the maximum degree of the graph is at most six.

Restricted instances

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Complete, bipartite item graphs

For a symmetric GMP instance, if the underlying item graph is a complete, bipartite graph $H = (U, V, U \times V)$ with U and V being the 2 partitions, a 2-stable allocation can be computed in $O(n^{\min(|U|,|V|)+4})$.

General instance

Existence

Observation. A 2-stable allocation may not always exists.

Theorem. Deciding the existence of a 2-stable allocation is NP-complete.

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Restricted instances

- Some structures where stable allocations exist and can be efficiently computed.
  - Core stable: agent graph is acyclic.
  - 2-stable: restricted cyclic agent graph with positive weights.

Solution. Variants of serial dictatorship.

Envy-freeness

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Swap envy-freeness

An allocation π is swap envy-free if it has no agents i, j such that $u_i(\pi') > u_i(\pi)$ where $\pi'(i) = \pi(j), \pi'(j) = \pi(i)$ and $\pi'(k) = \pi(k)\ \forall k \in N - \{i, j\}$.

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Computation

Theorem. For GMP with uniform valuation where the underlying player graph is an unweighted cycle, deciding if there exists a swap envy-free allocation is NP-complete.

Theorem. For GMP with symmetric neighbourhood, deciding the existence of a swap envy-free allocation is NP-complete.

Conclusion

A model for resource allocation which incorporates agents' influence within a local neighbourhood structure.

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Questions

- Analysis of stability based on complexity measures on graphs.
- Core-stability in symmetric neighbourhood.
- Other forms of externalities that have compact representations.
- Quality of stable outcomes.
- Fairness notions including approximations of envy-free allocations.