# The Sorites paradox, Fuzzy Logic, and Wittgenstein's Language Games

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## Abstract

We look at the history of research on vagueness and the Sorites paradox. That search has been largely unsuccessful and the existing solutions are not quite adequate. But following Wittgenstein we show that the notion of a *successful language game* works.

Language games involving words like “small” or “red” can be successful and people can use these words to cooperate with others. And yet, ultimately these words do not *have* a meaning in the sense of a tight semantics. It is just that most of the time these games work. It works to say, “the light is green and we can go,” even though the color word “green” does not actually *have* a semantics.

According to Hindu scriptures, the evil king Hiranyakashyapu prayed to the god *Brahma* to grant him the following boon:

> *Grant me that I not die within any residence or outside any residence, during the daytime or at night, nor on the ground or in the sky. Grant me that my death not be brought by any being other than those created by you, nor by any weapon, nor by any human being or animal.*

Eventually, Hiranyakashyapu was killed by a creature (*Narasimha*) which was half man and half lion, not killed in day or night but at dusk, and not indoors nor outdoors but at the doorstep.

Evidently, Hiranyakashyapu did not realize that with ambiguous $A, B$, a conjunction $\neg A \wedge \neg B$ could be true even though $A \vee B$ is also true. He was not killed indoors and he was not killed outdoors even though he was killed "either indoor or outdoor". We will return to this point which is also discussed by Michael Dummett

> *But, now, consider a vague statement, for instance 'That is orange'. If the object pointed to is definitely orange, then of course the statement will be definitely true; if it is definitely some other colour, then the statement will be definitely false; but the object may be a borderline case, and then the statement will be neither definitely true nor definitely false.... - the disjunctive statement, 'That is either orange or red' will be definitely true even though neither of its disjuncts is.*

(Dummett 1975)

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# Background

The Megarian philosopher Eubulides (4th century BC) is usually credited with the first formulation of the following puzzle.

1. 1 grain of wheat does not make a heap.
2. if n grains do not make a heap then n+1 grains do not make a heap. Therefore,
3. 1 million grains don't make a heap.

This inductive argument can be replaced by a large number of applications of *modus ponens*

1. 1 grain of wheat does not make a heap.
2. If 1 grain doesn't make a heap, then 2 grains don't.
3. If 2 grains don't make a heap, then 3 grains don't.
   ...
4. If 999,999 grains don't make a heap, then 1 million grains don't. Therefore,
5. 1 million grains don't make a heap.

Since *soros* is the Greek word for a heap, this puzzle is often known as the *Sorites paradox*. Here 2-4 replaced the inductive step, *if n grains do not make a heap then n+1 grains do not make a heap.*

There is a similar Indian story about a woman who cured her husband of his opium addiction. She weighed his usual opium ration against a ball of woolen thread. Then every day, she cut off about an inch or so of the ball and she weighed his opium ration against the reduced ball. The husband did not notice any difference from one day to the next, but over time, the ball became empty and the husband was cured.

This story must be recent as opium was only introduced into India by the Mughals in the 17th century, long after Eubulides.

# Precisification and super-truth

One way to deal with this problem is epistemic. Namely that there is an $n$ for which $n$ grains do not make a heap but that $n+1$ does but we do not know which $n$ it is. We will say that a statement is super-true if it is true regardless of which $n$ it is and it is super-false if it is false regardless of which $n$ it is. In that case $(\exists n)(\neg H(n) \wedge H(n+1))$ is super-true although we are unable to give an explicit $n$.

Kit Fine (Fine 75) says,

> *In this section we shall argue for the super-truth theory, that a vague sentence is true if it is true for all admissible and complete specifications. An intensional version of the theory is that a sentence is true if it is true for all ways of making it completely precise ... As such, it is a sort of principle of non-pedantry : truth is secured if it does not turn upon what one means.*

Fine manages to save a great deal of classical logic and our intuitions this way. Suppose there is doubt whether a certain patch is pink or red. Then, according to him, it is not (super)true that it is pink, it is not true that it is red, but it is true that it is either pink or red. For regardless of where we draw the boundary, it will be one or the other. And it is false that it is both pink and red for no path to a complete specification will render it both.

# Fuzzy logic

Lotfi Zadeh (1965) addresses this problem by resorting to truth values properly between 0 and 1.

> *More often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership. For example, the class of animals clearly includes dogs, horses, birds, etc. as its members, and clearly excludes such objects as rocks, fluids, plants, etc. However, such objects as starfish, bacteria, etc. have an ambiguous status with respect to the class of animals. The same kind of ambiguity arises in the case of a nmnber such as 10 in relation to the "class" of all real numbers which are much greater than 1."*

# Fuzzy logic, contd

*Clearly, the "class of all real numbers which are much greater than 1," or "the class of beautiful women," or "the class of tall men," do not constitute classes or sets in the usual mathematical sense of these terms. Yet, the fact remains that such imprecisely defined "classes" play an important role in human thinking, particularly in the domains of pattern recognition, communication of information, and abstraction.*

Zadeh addresses his concerns by allowing fuzzy truth values in the real interval [0,1]. Let $R(x)$ indicate that the object $x$ is red. If the fuzzy truth value of $R(x)$ is close to zero then it means that $x$ is pretty much not red. If it is close to 1 then $x$ is pretty close to being red. Now if you have a series of objects $o_1, o_2, ..., o_{100}$ and the $R$ values gradually go up from 0 to 1 then one could say that the objects are gradually becoming more red, although you might not be able to distinguish say $o_{49}$ and $o_{50}$. Thus the Sorites paradox is defanged so to say.

One would expect that even though fuzzy truth values lie between 0 and 1 they should at least be *interpersonal*. If I think that something is .3 red then you should also think that it is .3 red. But in a test I administered in Sicily, (Parikh 1991), I found that people gave very different fuzzy values to questions like “is a handkerchief an item of clothing?” or “is Sonia Gandhi an Indian?”. A handkerchief is made of cotton but we do not usually wear it so to the first question, intuitions conflict. Similarly Sonia Gandhi was Italian by birth but became an Indian citizen so again intuitions conflicted. These conflicts were resolved by different people in different ways so there was no stable fuzzy value.

# Almost consistent theories

But perhaps communication does not necessitate having the exact same truth value for a proposition and perhaps *some* agreement suffices and is useful. We will return to this question.
Let us define an *inductive* set of natural numbers to be a nonempty set $X$ such that if $n \in X$ then $n + 1 \in X$. Clearly such a set will contain all large numbers and if it contains 0, it will contain all numbers.
Let us define a *bounded* set $X$ to be a set such that $(\exists M)(\forall n)(n \in X \rightarrow n < M)$. Clearly a bounded set must be finite whereas an inductive set will be infinite. So can a set be both bounded and inductive? Seems not.

Yesenin-Volpin points out that such sets do exist in some sense. For let $H$ be the set of heartbeats in one's childhood. No one ceases to be a child in a single heartbeat. So if $n \in H$ then $n + 1 \in H$. And yet, assuming at most a hundred heartbeats per minute, there are fewer than 10 million heartbeats before one reaches the age of eighteen. So $H$ is bounded above by 10 million. $H$ is both inductive and bounded.

But isn't it inconsistent to speak of $H$ at all?

To make things easier let us replace 10 million by a mere 100. The inconsistency should be even more glaring.

Consider a set $X$ of formulas $\{H(1), H(1) \rightarrow H(2), ...H(99) \rightarrow H(100), \neg H(100)\}$ This set is inconsistent as the conclusion $H(100)$ can be derived from the first 100 formulas in $X$ contradicting the last formula.

However, let $T$ be some consistent theory, say $T$ is PA = Peano Arithmetic. Let $T' = T \cup X$. Evidently $T'$ is inconsistent since it includes $X$. However, let $A$ be some formula of number theory (not involving the predicate $H$) whose proof in $T'$ takes less than 100 lines. Then $A$ is a theorem of PA.

**Proof:** Clearly the proof includes less than 100 formulas of the form $H(n) \to H(n+1)$ and hence some formula of the form $H(n) \to H(n+1)$ is absent. To fix thoughts suppose that the formula $H(50) \to H(51)$ is missing from the proof. Then extend the usual interpretation of PA by interpreting $H(k)$ as $k < 51$. All the formulas which *occur* in the proof become true and $A$ becomes a theorem of PA. $\square$

This means that even though, in the classical sense there is no set like $H$ and the properties of the putative $H$ are inconsistent, it still *works* to reason with $H$. Only a dogmatic person will insist that there are no children or no heartbeats in childhood "because the very notions are inconsistent".

# Vagueness and communication

In the following example, the usefulness of communication consists of a saving of time. Ann and Bob teach at the same college. Ann teaches Math and Bob teaches History. One day Ann telephones Bob from school.

Ann: Bob, can you bring my topology book in?
Bob: What does it look like?
Ann: It is blue.
Bob: OK.
Ann: Be sure to bring it, I am going to lunch now, but I need it for class at 2 PM.

It so happens that Ann and Bob have somewhat different notions of what the word “blue” means, i.e. which things it applies to.

Among Ann's 1,000 books, there are 250 that Ann would call blue whereas there are 300 that Bob would call blue.

Let X= Blue(Ann), the set of those books that Ann considers to be blue, and Y = Blue(Bob), the set of those books that Bob considers to be blue.

There are 225 books that both would call blue (i.e. they are in $X \cap Y$)

675 that neither would (they are in $\overline{X} \cap \overline{Y}$).

But there are 25 books that Ann, but not Bob would call blue ((they are in $X \cap \overline{Y}$))

and 75 books that Bob, but not Ann, would call blue (they are in $\overline{X} \cap Y$). So they would, if asked, disagree on 100 books.

I shall assume that neither Ann nor Bob is aware of this. Now Ann intends Bob to look through the set X, but having his own notion of what blue is, he will look in Y = Blue(Bob). Here are the expected (average) number of books that Bob would look at in two cases.

*If Bob had no information:* 500 books on average. (If he is lucky, the first book that he looks at will be the topology book. If unlucky, the last book he looks at will be the topology book. The average is 500.5, or approximately 500.)

*In the actual case*, since the book is in X, with probability 0.9 it is also in Y. Since Bob actually looks in Y, and if necessary, in the complement of Y, with probability 0.9 he only needs to look through Y, or at most 300 books, yielding 150 average. With probability 0.1 he will not find it in Y.

In time he will have looked through all of Y and will only need to look at about 350 further books in the complement of Y. Thus he will look at $.9(150) + .1(300 + 350) = 200$ books, which is the average in this case.

Thus Bob is saved considerable labour by what Ann said though his interpretation of "blue" is not what Ann intended. Instead of having to look through 500 he looks through 200. No *proposition* is conveyed by Ann to Bob for they do not share a semantics for *blue* but he is helped.

Here is Wittgenstein in his *Remarks on the Foundations of Mathematics*

> *"What we call counting is an important part of life's activities. Counting and calculating are not – e.g. – simply a pastime....*
>
> *The truth is that counting has proved to pay – "then do you want to say that being true means being usable or useful?" "No, not that but that it can't be said of the series of natural numbers anymore than of our language that it is true, but that is usable, and above all* it is used"*.*

Coming back to Ann, we need not ask if Ann was speaking the truth when she said that the book was blue. There is indeed a 10% chance that Bob would disagree with her. But she did *help* Bob in his search for her book. I think Wittgenstein would like this example where a language game is successful even though it is not underpinned by a solid notion of objective truth.

I would suggest that the reason this problem has been so thorny is that we have been looking for a semantics and a logic. We did not consider that there might be *successful* language games without there being a semantics to justify our language.

# Vagueness and language games

Suppose that a community of people (like us) use words like "blue", "red", "large", "small" etc. and assign certain properties to the putative predicates. But on second thought it turns out that these properties are inconsistent and hence there *are* no such predicates (classically speaking). Does it follow that these people should constantly fall into confusion and be perpetually at war with each other? Not so.

For we saw that Bob and Ann can "communicate" even though they assign different extensions to the word "blue". We also saw that someone who believes that (i) 0 is small, that (ii) if $n$ is small then so is $n + 1$ and that (iii) $10^{10}$ is not small, may succeed in making correct inferences provided only that she does not perform deductions of more than $10^{10}$ lines.

# Dealing with non-transitivity

If we define $I(x, y)$ to mean that $x, y$ are indiscriminable in some important way then $I$ is reflexive and symmetric but may not not transitive. In other words, there can exist $x, y, z$ such that $x, y$ are indiscriminable, $y, z$ are indiscriminable, but $x, z$ are discriminable. Thus $I$ not an equivalence relation. This fact is of course behind the Sorites paradox. This can create a problem in practical matters as when we are sorting socks after a wash and dry.

Suppose we have six socks, A, B, C, D, E, F where the sets {A,B}, {C,D} and {E,F} are respectively from three different pairs of socks. Moreover each of A, B will match each of C, D. Each of C, D will match with each of E, F. However, because of intransitivity, A, B do not match E, F.

A ... B ... C ... D ... E ... F

Suppose now that all six socks have been washed and dried and, relying on matching, we pair together B,C which match. We also put together D, E which match.

A ... (B ... C) ... (D ... E) ... F

We are now left with A, F which do *not* match! How do we deal with this problem? We relied on indiscriminability which is not transitive. We could start over, but if there are a lot of socks we might be working for ever!

At first sight it looks as if finding a good matching might be an NP-complete problem, quite hard if there are a hundred socks. *No doubt this is one reason centipedes do not wear socks!*
It turns out that there *is* a transitive relation $J$ which depends on $I$, indiscriminability, but does not coincide with it. Given a sock $s$, let $M(s) = \{t : I(s,t)\}$ And let $J(s,t)$ mean that $M(s) = M(t)$. Then $J(s,t)$ implies $I(s,t)$ but is stronger. Moreover, $J$ is transitive. Relying on $I$, we construct $J$ and pair two socks $s, t$ iff $J(s,t)$. This algorithm runs in $n^2$ time, showing that the original problem was not NP-complete.

To see that $J(x, y)$ implies $I(x, y)$ note the following. Suppose $J(x, y)$. Now $I(x, x)$ holds. Hence $x \in M(x)$. Given $J(x, y)$, $M(x) = M(y)$ and hence $x \in M(y)$. Ergo $I(x, y)$.
On the other hand $I$ does not imply $J$, for in the example with A,..,F above, $I(B, C)$ holds. But while $I(C, E)$ holds, $I(B, E)$ fails. Hence $E \in M(C)$ but $E \notin M(B)$. So $J(B, C)$ fails, and $I(B, C)$ fails to imply $J(B, C)$. (See Parikh et al 2001 for details).

However, we do need a theory of what happens when *different agents* interpret the same vague predicate in different ways. Many cooks making incompatible decisions can spoil a broth, but they could also come up with a good feast. Vagueness is not always a disaster.

Consider the following situation. Country A moves some troops to its border with country B. If there is one soldier, we will not say, "A is massing troops," and if there are a million, we will. So "A is massing troops on the border with B" is a vague statement and interpreted by different governments and different generals in different ways. And yet we can predict *something*. We badly need a theory of how that happens.

# Conclusion

We showed quite convincingly that vague predicates do not have a semantics and hence they do not have a logic. But they do have a *use* and we found how this use falls inside Wittgenstein's requirement that a language game be useful and be used.

Here is a question I would raise – for the future. Suppose that people's reactions to "is it blue?" go according to experimental data. Then they will have different "semantics" for blue which will vary a little from person to person and from the same person to himself from time to time. But some algorithms will still "work". It is perfectly fine to say, "green means go and red means stop" even though both red and green are vague predicates.

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