Persuasion with limited communication capacity

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joint work with **Tristan Tomala** (HEC Paris, GREGHEC UMR 2959)

”ReLaX” Workshop on Games - Chennai Mathematical Institute
3rd February 2021

Overview

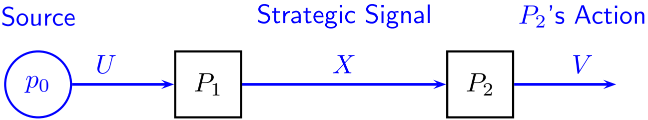
- Autonomous Devices with **Non-Aligned** Objectives/Utilities

- Transmission of Strategic Information in **Economics** Literature
  → **Bayesian Persuasion** : [Kamenica Gentzkow (AER) 2011]

- We characterize the solution when the **communication channel is noisy**
  → [Le Treust and Tomala, Journal of Economic Theory, Vol. 184, Nov. 2019]
  https ://arxiv.org/abs/1711.04474

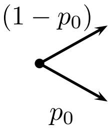
- We extend the solution by considering **decoder side information**
  → [Le Treust and Tomala, draft 2018]
  https ://arxiv.org/abs/1807.05147

Bayesian Persuasion [Kamenica Gentzkow (AER) 2011]



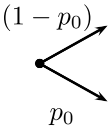
Distinct Utility Functions

$\phi_1(u,v)$



$$\begin{array}{c|c|c|}
 & v_0 & v_1 \\ \hline
u_0 & 0 & 1 \\ \hline
u_1 & 0 & 1 \\ \hline
\end{array}$$

$\phi_2(u,v)$



$$\begin{array}{c|c|c|}
 & v_0 & v_1 \\ \hline
u_0 & 1 & 0 \\ \hline
u_1 & 0 & 1 \\ \hline
\end{array}$$

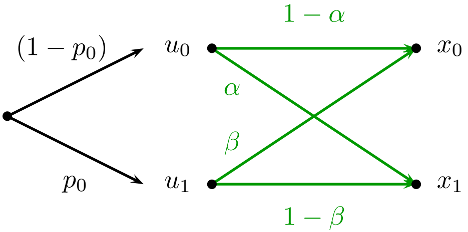
P_2 : **Decision Maker/Decoder**

P_1 : **Persuader/Encoder**

1. P_1 chooses/announces $\mathbb{P}(x|u)$
2. (U,X) is drawn with $\mathbb{P}(u) \times \mathbb{P}(x|u)$
3. P_2 observes X and chooses V
4. Player k's payoff is $\phi_k(u,v)$

Control of the posterior beliefs

P_1's signaling strategy : $\mathbb{P}(x|u)$ with $\alpha \in [0,1]$, $\beta \in [0,1]$



Posterior Distributions (p_1, p_2) :

$$\mathbb{P}(u_1|x_0) = \frac{p_0 \cdot \beta}{p_0 \cdot \beta + (1-p_0) \cdot (1-\alpha)} = p_1$$

$$\mathbb{P}(u_1|x_1) = \frac{p_0 \cdot (1-\beta)}{p_0 \cdot (1-\beta) + (1-p_0) \cdot \alpha} = p_2$$

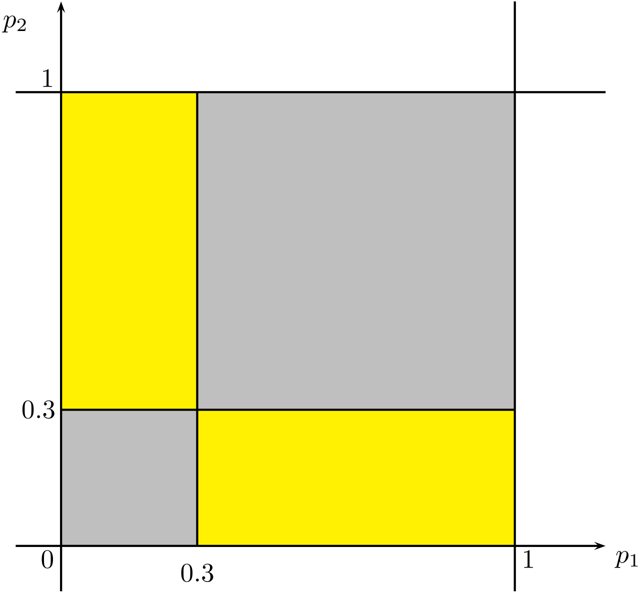
$p_0 = \mathbb{P}(u_1)$: Prior Distribution

Given any prior p_0 and posteriors (p_1, p_2), the signaling parameters (α, β) are :

$$\begin{aligned} \alpha &= \frac{(1-p_2) \cdot (p_0 - p_1)}{(1-p_0) \cdot (p_2 - p_1)} \\ \beta &= \frac{p_1 \cdot (p_2 - p_0)}{p_0 \cdot (p_2 - p_1)} \end{aligned}$$

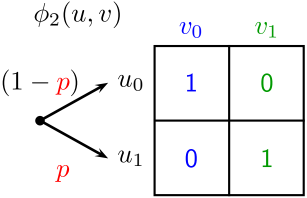
Under condition : $0 \leq p_1 \leq p_0 \leq p_2 \leq 1$ or $0 \leq p_2 \leq p_0 \leq p_1 \leq 1$

Region of Posterior Beliefs (p_1, p_2) for prior $p_0 = 0.3$



Best-Response $BR_2(p)$ of P_2 given the belief p

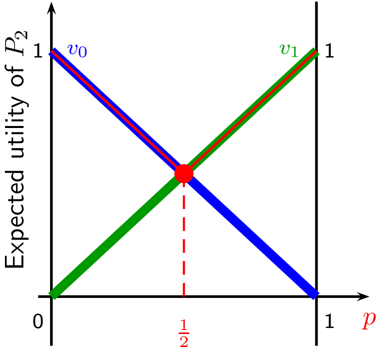
- Denote by $p = \mathbb{P}(u_1)$ the **belief** of P_2



$(1-p)$

p

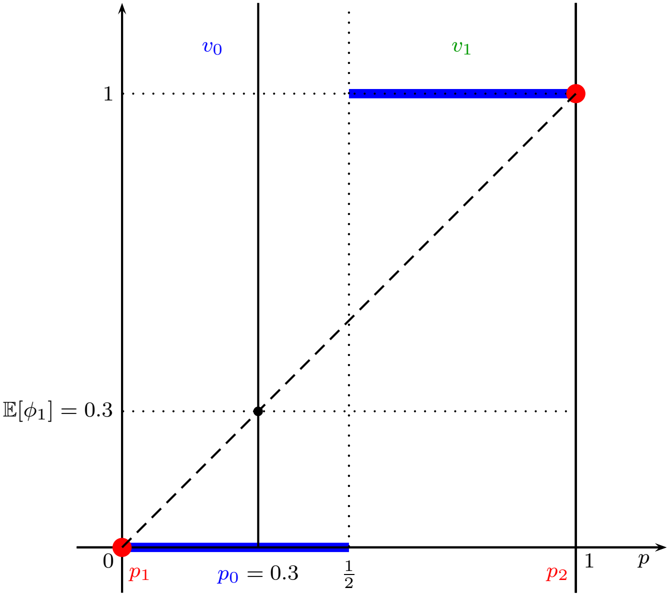
$\phi_2(u,v)$	v_0	v_1
u_0	1	0
u_1	0	1



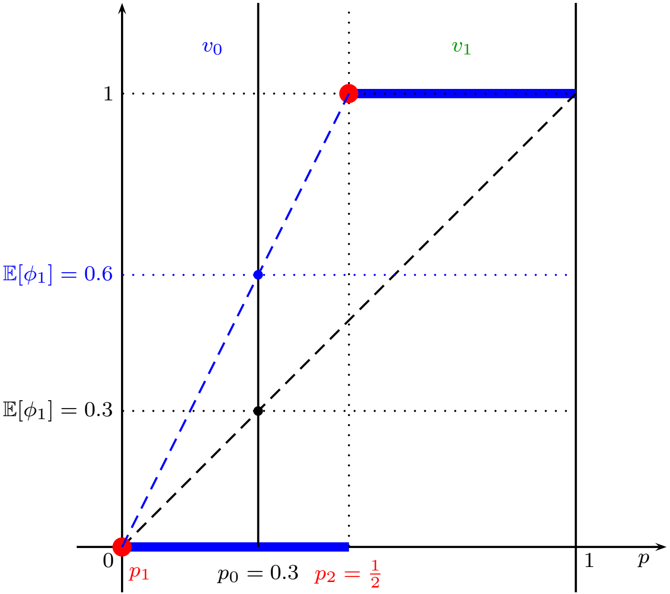
$BR_2(p)$ Best-Response of P_2 depending on his belief p

- Player P_2 plays v_1 iff his **belief** $p \geq \frac{1}{2}$
- Player P_1 wants player P_2 to play v_1
- Player P_1 choose the signalling $\mathbb{P}(x|u)$

Utility of P_1 and full revelation



Full revelation is not optimal for P_1



Characterization

Given decoder's belief $\mathbb{P}(u)$, the Best-Reply $v^\star$ is :

$$v^\star \in \operatorname{argmax} \sum_u \mathbb{P}(u)\phi_2(u,v) = BR_2(\mathbb{P}(u))$$

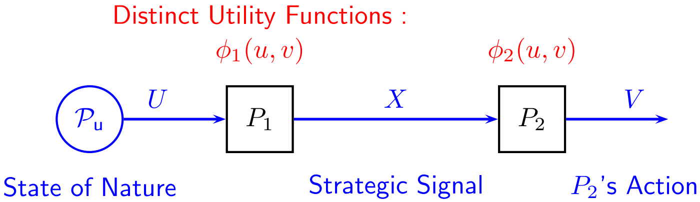
and the sender's payoff is : $\Psi\big(\mathbb{P}(u)\big) = \sum_u \mathbb{P}(u)\phi_1(u, v^\star)$.

Concavification [Kamenica Gentzkow 2011]

$$\begin{aligned} \Phi_1^\star \quad &= \quad \sup \sum_w \mathbb{P}(w)\Psi\big(\mathbb{P}(U|w)\big) \\ \text{s.t.} \quad & \qquad \sum_w \mathbb{P}(w)\mathbb{P}(u|w) = \mathcal{P}(u) \end{aligned}$$

This solution relies on Splitting Lemma [Aumann Maschler 1995]

Persuasion with noisy communication ?



Distinct Utility Functions :

$\phi_1(u,v)$ $\phi_2(u,v)$

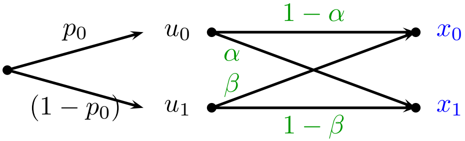
$\mathcal{P}_{\mathsf{u}}$ $\xrightarrow{U}$ P_1 $\xrightarrow{X}$ P_2 $\xrightarrow{V}$

State of Nature

Strategic Signal

P_2's Action

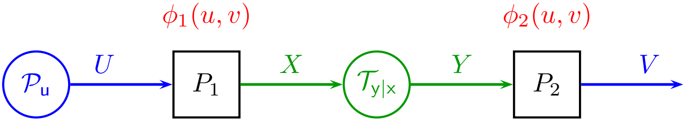
- Player P_1 sends a strategic signal X in order to control posteriors (p_1, p_2),
- Player P_2 implements a best-reply to her belief $p \in [0,1]$



$$\mathcal{P}(u_0|x_0) = \frac{p_0 \cdot (1-\alpha)}{p_0 \cdot (1-\alpha) + (1-p_0) \cdot \beta} = p_1$$

$$\mathcal{P}(u_0|x_1) = \frac{p_0 \cdot \alpha}{p_0 \cdot \alpha + (1-p_0) \cdot (1-\beta)} = p_2$$

Persuasion with noisy communication ?



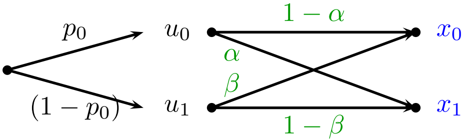
Distinct Utility Functions :

State of Nature

Noisy Channel $\mathcal{T}(y|x)$

P_2's Action

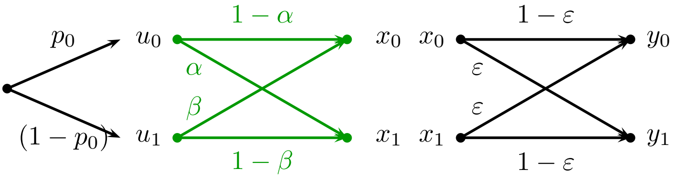
- Player P_1 sends a strategic signal X in order to control posteriors (p_1, p_2),
- Player P_2 implements a best-reply to her belief $p \in [0,1]$

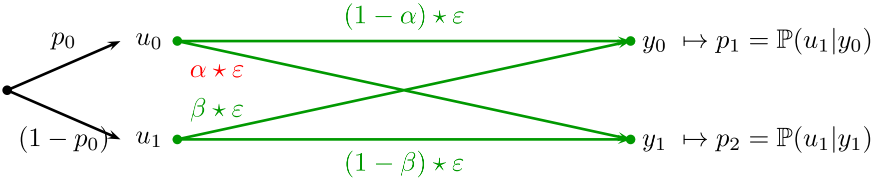


$$\mathcal{P}(u_0|x_0) = \frac{p_0 \cdot (1-\alpha)}{p_0 \cdot (1-\alpha) + (1-p_0) \cdot \beta} = p_1$$

$$\mathcal{P}(u_0|x_1) = \frac{p_0 \cdot \alpha}{p_0 \cdot \alpha + (1-p_0) \cdot (1-\beta)} = p_2$$

One-shot Persuasion with Noisy Channel $\varepsilon \in [0, 0.5]$





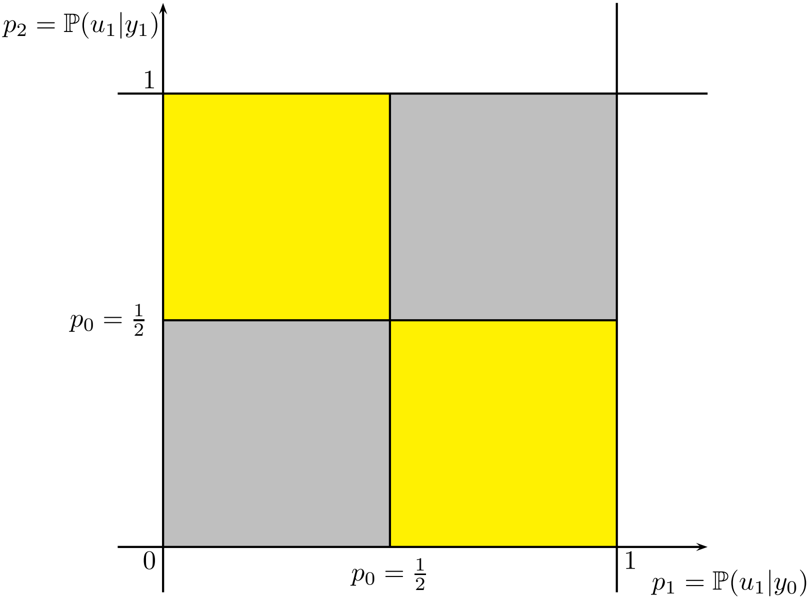
$$\alpha\star\varepsilon = \alpha\cdot(1-\varepsilon)+(1-\alpha)\cdot\varepsilon \quad \in \quad [\varepsilon, 1-\varepsilon]$$

Lemma MLT and Tomala - IEEE Allerton Conference 2016

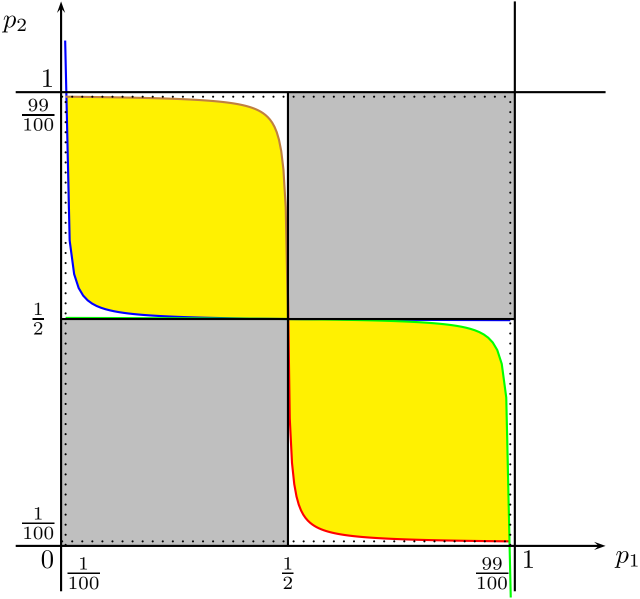
Posterior distributions (p_1, p_2) *are achievable if and only if* $\exists\,(\alpha,\beta)$ *s.t. :*

$$\alpha\star\varepsilon = \frac{(1-p_2)\cdot(p_0-p_1)}{(1-p_0)\cdot(p_2-p_1)}, \qquad \beta\star\varepsilon = \frac{p_1\cdot(p_2-p_0)}{p_0\cdot(p_2-p_1)}.$$

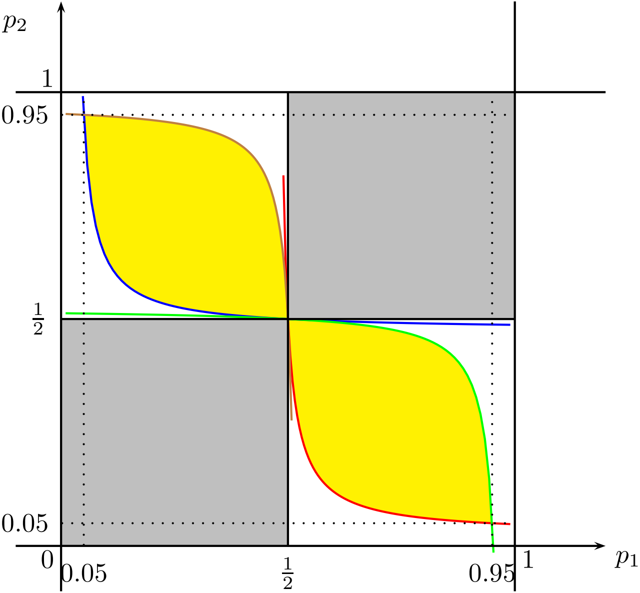
Region of posteriors (p_1, p_2) without channel noise $\varepsilon = 0$



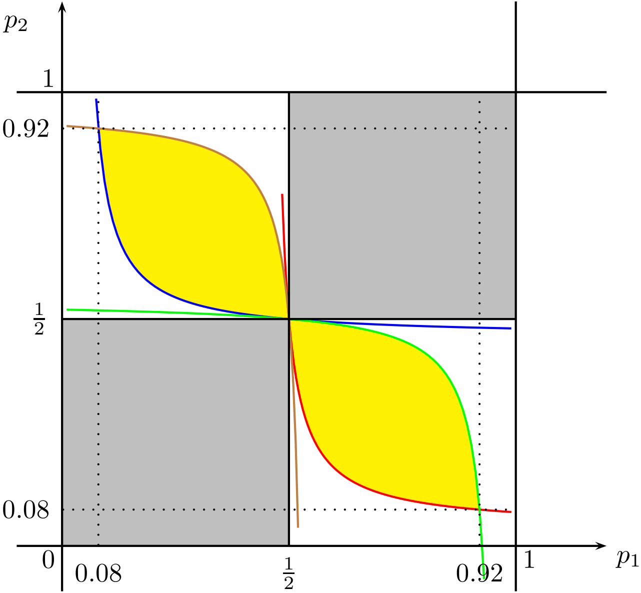
Region of posteriors (p_1, p_2) for channel noise $\varepsilon = 0.01$



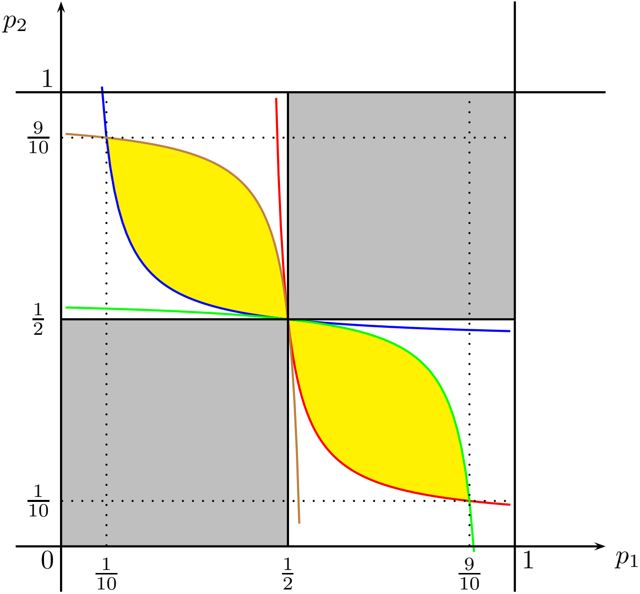
Region of posteriors (p_1, p_2) for channel noise $\varepsilon = 0.05$



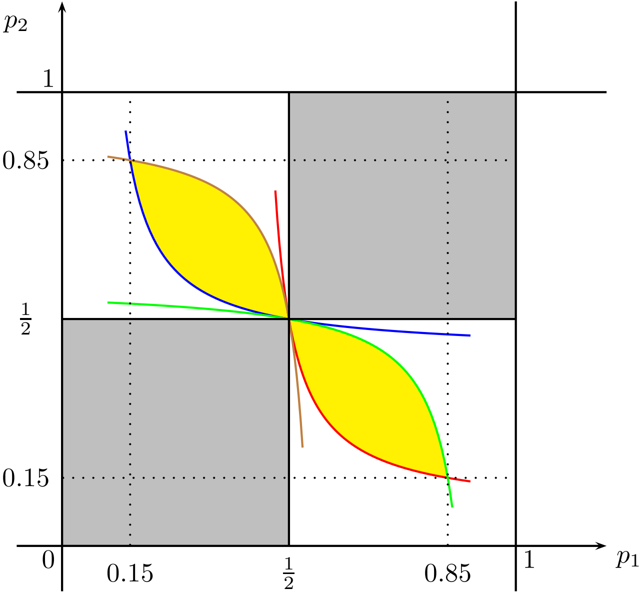
Region of posteriors (p_1, p_2) for channel noise $\varepsilon = 0.08$



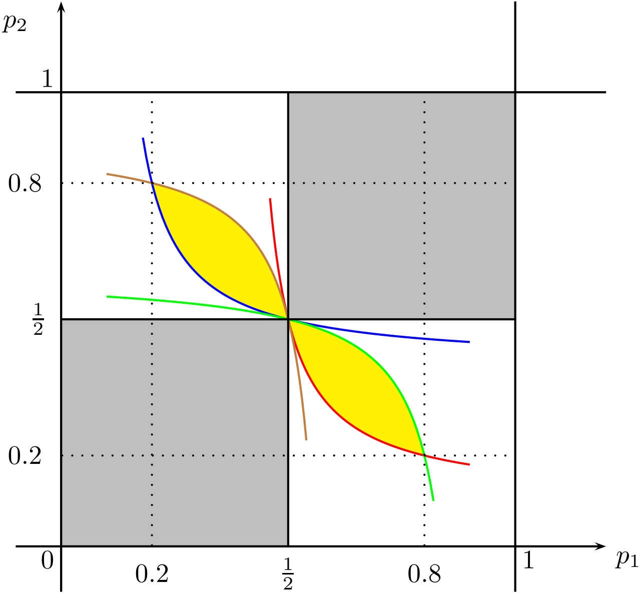
Region of posteriors (p_1, p_2) for channel noise $\varepsilon = 0.1$



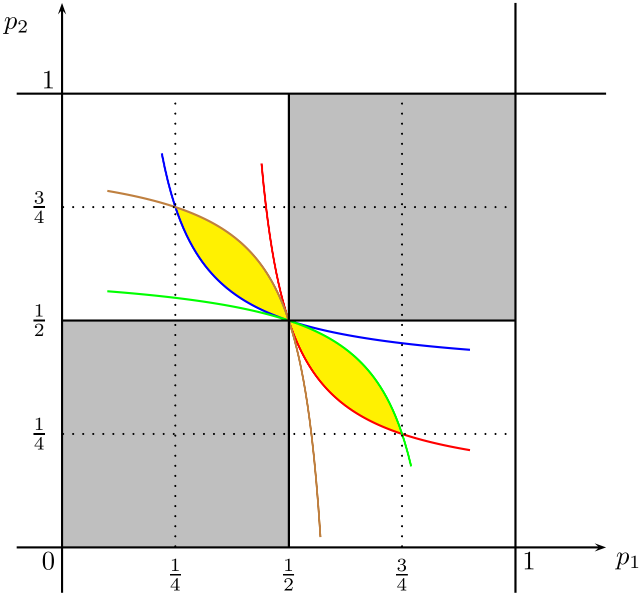
Region of posteriors (p_1, p_2) for channel noise $\varepsilon = 0.15$



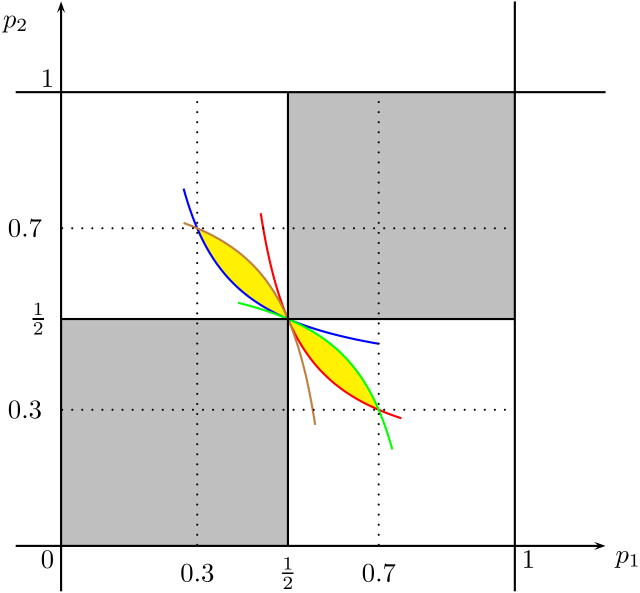
Region of posteriors (p_1, p_2) for channel noise $\varepsilon = 0.20$



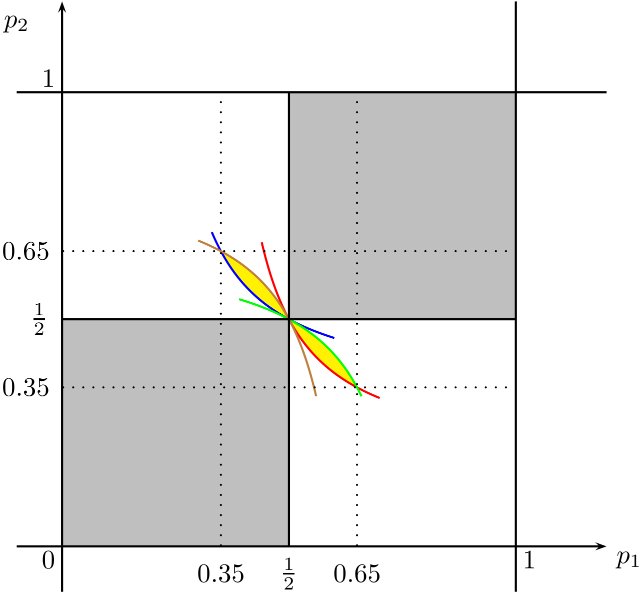
Region of posteriors (p_1, p_2) for channel noise $\varepsilon = 0.25$



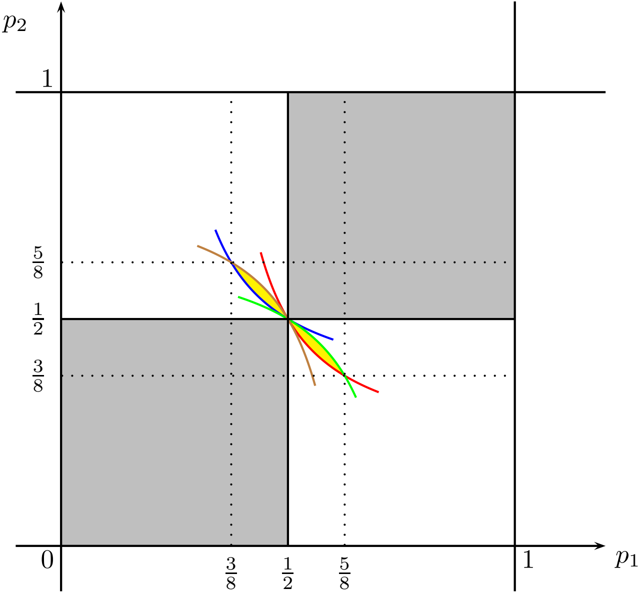
Region of posteriors (p_1, p_2) for channel noise $\varepsilon = 0.30$



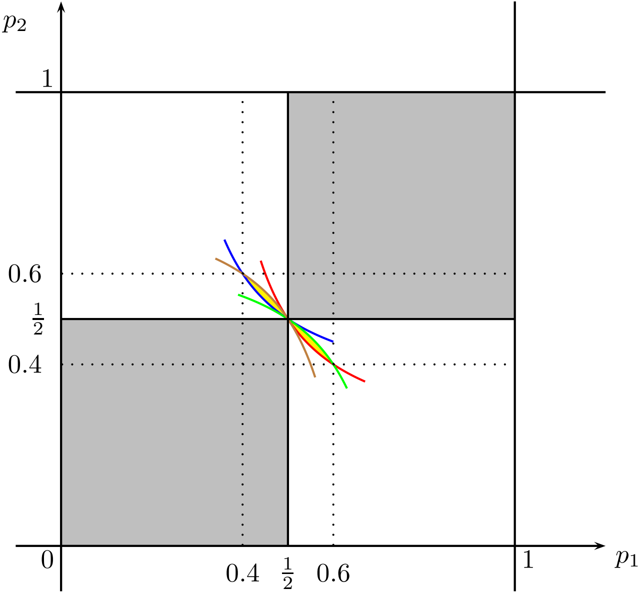
Region of posteriors (p_1, p_2) for channel noise $\varepsilon = 0.35$



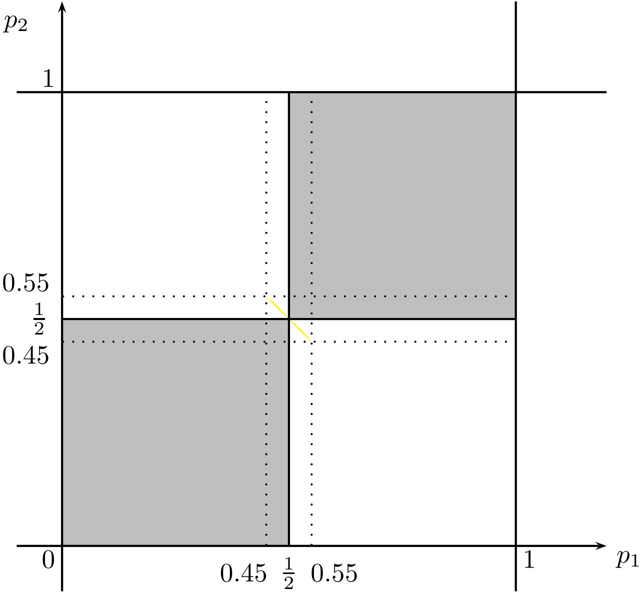
Region of posteriors (p_1, p_2) for channel noise $\varepsilon = 0.375$



Region of posteriors (p_1, p_2) for channel noise $\varepsilon = 0.40$



Region of posteriors (p_1, p_2) for channel noise $\varepsilon = 0.45$



[Le Treust and Tomala, 2019]

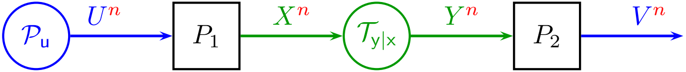
"Persuasion with limited communication capacity"

Journal of Economic Theory, Vol. 184, pp. 104940, Nov. 2019
https ://arxiv.org/abs/1711.04474

Joint source-channel coding

Long-run

Utility Functions : $\Phi_k(\sigma,\tau) = \mathbb{E}_{\sigma,\tau}\left[\frac{1}{n}\sum_{i=1}^{n}\phi_k(u_i,v_i)\right]$ for player $k \in \{1,2\}$



Source i.i.d. Noisy Channel i.i.d. P_2's Action

Player P_1's Strategy : $\sigma : \mathcal{U}^n \to \Delta(\mathcal{X}^n)$

Player P_2's Strategy : $\tau : \mathcal{Y}^n \to \mathcal{V}^n$

- Player P_1 chooses and announces strategy $\sigma(x^n|u^n)$ (commitment power).
- (U^n, X^n, Y^n) are drawn with $\prod_{i=1}^{n}\mathcal{P}_{\mathsf{u}}(u_i) \times \sigma(x^n|u^n) \times \prod_{i=1}^{n}\mathcal{T}(y_i|x_i) = \mathbb{P}_\sigma$
- Sequence Y^n is announced to P_2.
- Player P_2 chooses a sequence of actions with $v^n = \tau(y^n)$.

Information Theory Tools

The entropy $H(U)$ and the mutual information $I(U;W)$ are defined by :

$$H(U) = \mathbb{E}_u\Big[\log_2 \frac{1}{p(u)}\Big] = \sum_u p(u)\log_2 \frac{1}{p(u)}$$

$$I(U;W) = \mathbb{E}_{uw}\Big[\log_2 \frac{p(u,w)}{p(u)p(w)}\Big] = \sum_{uw} p(u,w)\log_2 \frac{p(u,w)}{p(u)p(w)}$$

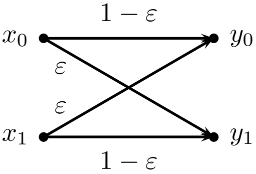
The capacity C of a noisy channel $\mathcal{T}(y|x)$ is defined by :

$$C = \max_{p(x)\in\Delta(\mathcal{X})} I(X;Y) = \max_{p(x)\in\Delta(\mathcal{X})} \mathbb{E}\Big[\log_2 \frac{\mathcal{T}(y|x)}{\sum_x p(x)\cdot\mathcal{T}(y|x)}\Big]$$

Ex : $C = 1 - \Big(\varepsilon\cdot\log_2\frac{1}{\varepsilon} + (1-\varepsilon)\cdot\log_2\frac{1}{1-\varepsilon}\Big)$,

for $\varepsilon = 0.25$, $C \simeq 0.19$ bits,

$|M| = 2^C \simeq 1.14$: average number of messages correctly transmitted per channel use



Characterization

We define $\Psi\big(\mathbb{P}(u)\big)$ the robust payoff at belief $\mathbb{P}(u)$:

$$\Psi\big(\mathbb{P}(u)\big) = \min_{v \in \operatorname{argmax} \sum_u \mathbb{P}(u)\phi_2(u,v)} \sum_u \mathbb{P}(u)\phi_1(u,v)$$

Splitting with information constraint - reformulation of Allerton 2016

$$\begin{aligned} \Phi_1^\star \quad &= \quad \sup \sum_w \mathbb{P}(w)\Psi\big(\mathbb{P}(U|w)\big) \\ \text{s.t.} \quad & \quad \sum_w \mathbb{P}(w)\mathbb{P}(u|w) = \mathcal{P}(u), \\ \text{and} \quad & \quad \sum_w \mathbb{P}(w)H\Big(\mathbb{P}(U|w)\Big) \geq H(U) - C \end{aligned}$$

Information constraint $I(U;W) \leq C$ with auxiliary random variable W.

Main Result

- Player 2's Best Replies $BR_2(\sigma) = \arg\max_\tau \Phi_2(\sigma, \tau)$:

$$BR_2(\sigma) = \arg \max_{v^n = \tau(y^n)} \mathbb{E}_{\sigma,\tau}\left[\frac{1}{n}\sum_i^n \phi_2(u_i, v_i)\right]$$

Theorem MLT and Tomala, JET 2019

We characterize the best payoff player P_1 can secure :

$$1) \qquad \forall n \in \mathbb{N},\ \forall \sigma, \qquad \min_{\tau \in BR_2(\sigma)} \Phi_1(\sigma, \tau) \leq \Phi_1^\star,$$

$$2) \qquad \forall \varepsilon > 0,\ \exists \bar{n},\ \forall n \geq \bar{n},\ \exists \sigma, \qquad \min_{\tau \in BR_2(\sigma)} \Phi_1(\sigma, \tau) \geq \Phi_1^\star - \varepsilon.$$

Comments

Lemma MLT and Tomala, JET 2019

The optimal splitting for $\Phi_1^\star$ has a number of posteriors $|\mathcal{W}|$ restricted to :

$$|\mathcal{W}| = \min\Big(|\mathcal{V}|, |\mathcal{U}| + 1\Big).$$

1) Extend the mapping $\Psi\big(\mathbb{P}(u)\big)$ on the domain :

$$\mathcal{D} = \{(p,h) \in \Delta(\mathcal{U}) \times \mathbb{R} : 0 \leq h \leq H(U)\}$$

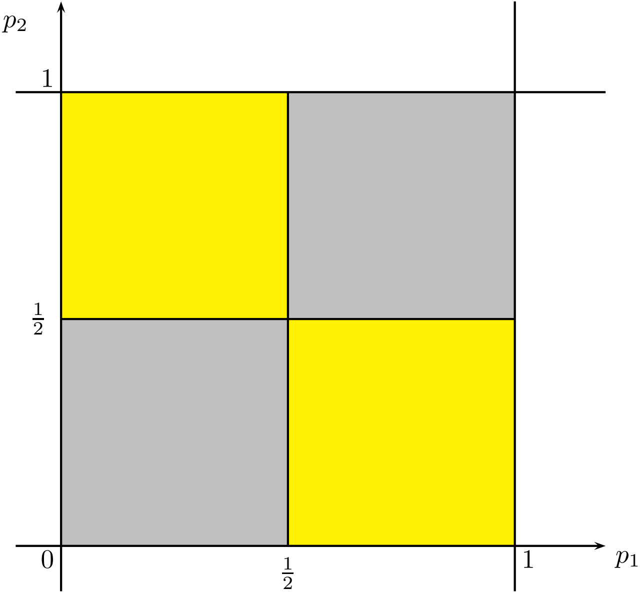
by $\Psi\big(\mathbb{P}(u), h\big) = \Psi\big(\mathbb{P}(u)\big)$. Then, $\Phi_1^\star = \mathsf{Cav}_{\,p,h}\,\Psi\Big(\mathcal{P}(u), H(U) - C\Big)$

2) Lagrangian of the concavification :

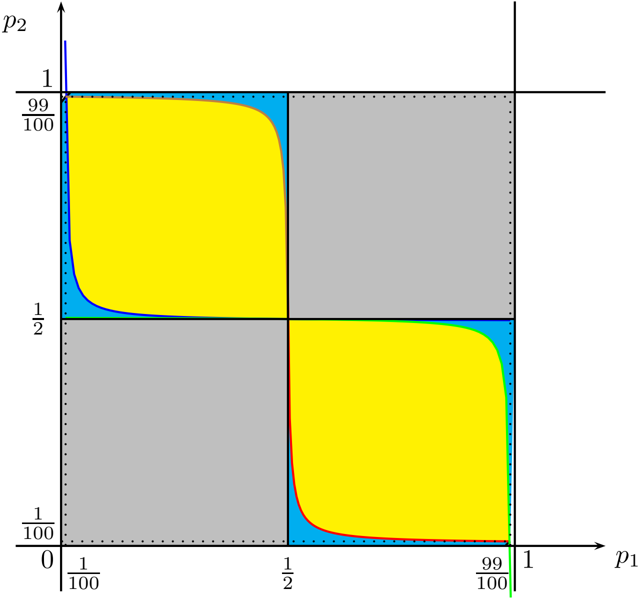
$$\Phi_1^\star = \inf_{t\geq 0}\Big\{\mathsf{Cav}\,(\Psi + tH)(p_0) - t(H(U) - C)\Big\}$$

related with the **Cost of Information** in [Kamenica Gentzkow (AER) 2014]

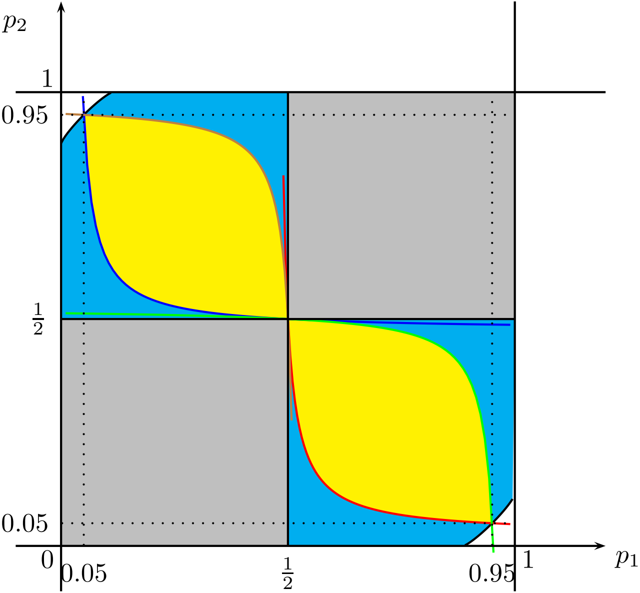
Region of Posteriors (p_1, p_2) without Noise $\varepsilon = 0$



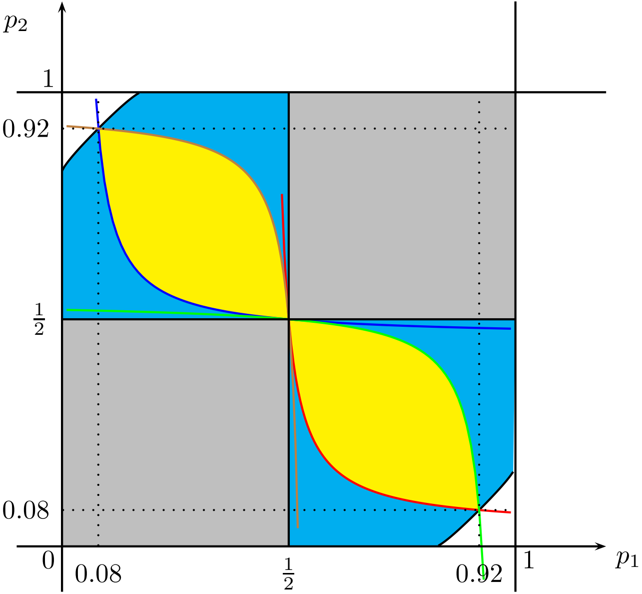
Region of Posteriors (p_1, p_2) for Channel Noise $\varepsilon = 0.01$



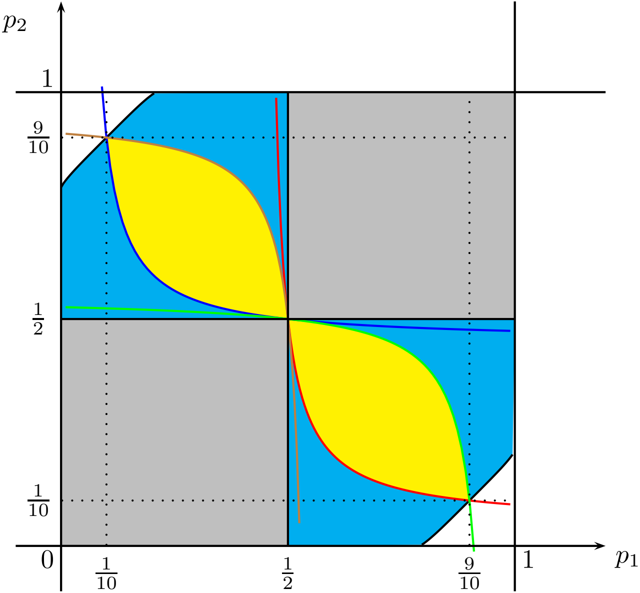
Region of Posteriors (p_1, p_2) for Channel Noise $\varepsilon = 0.05$



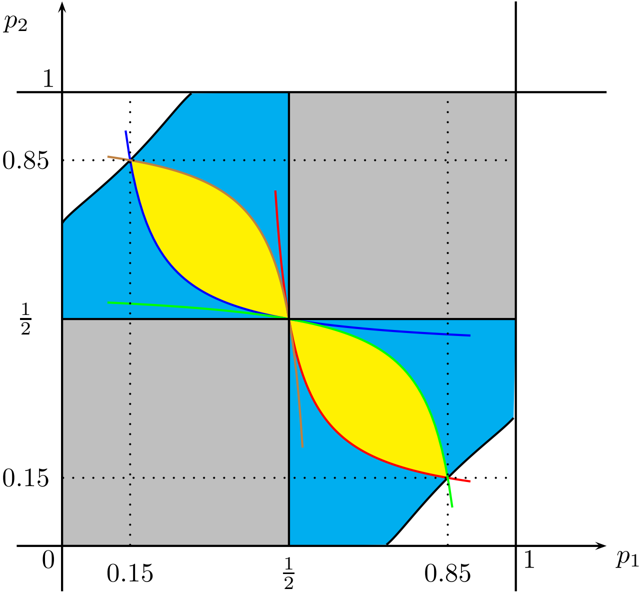
Region of Posteriors (p_1, p_2) for Channel Noise $\varepsilon = 0.08$



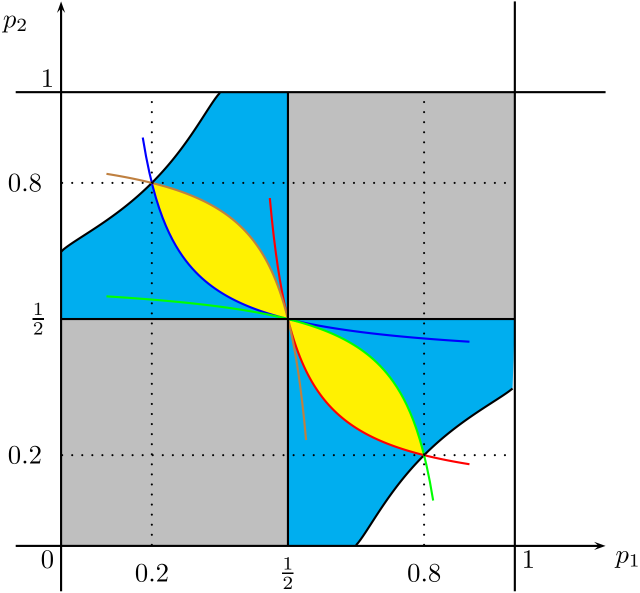
Region of Posteriors (p_1, p_2) for Channel Noise $\varepsilon = 0.1$



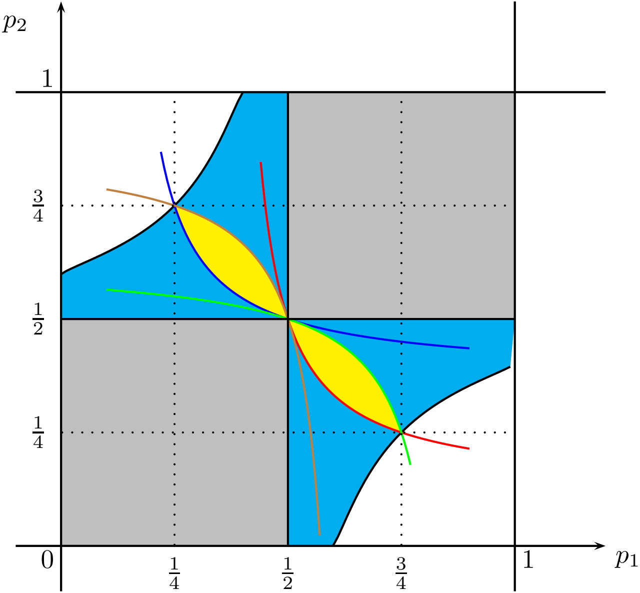
Region of Posteriors (p_1, p_2) for Channel Noise $\varepsilon = 0.15$



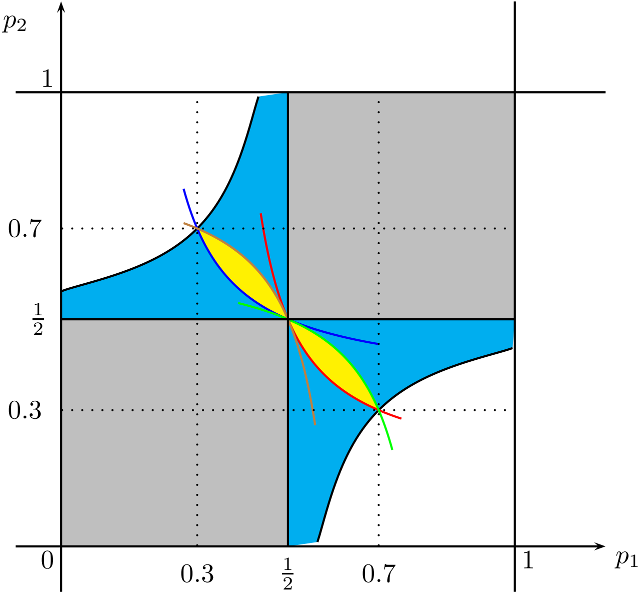
Region of Posteriors (p_1, p_2) for Channel Noise $\varepsilon = 0.20$



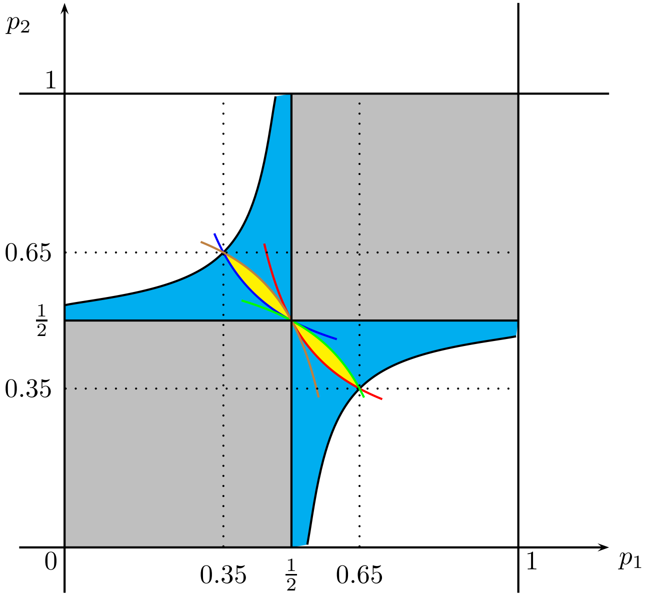
Region of Posteriors (p_1, p_2) for Channel Noise $\varepsilon = 0.25$



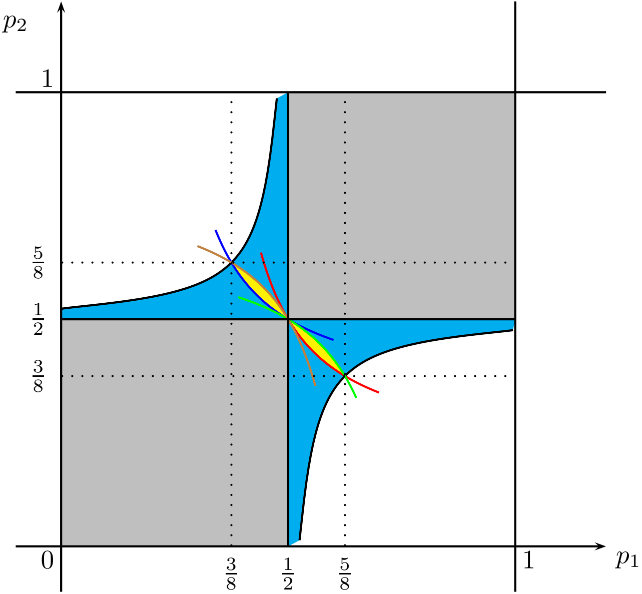
Region of Posteriors (p_1, p_2) for Channel Noise $\varepsilon = 0.30$



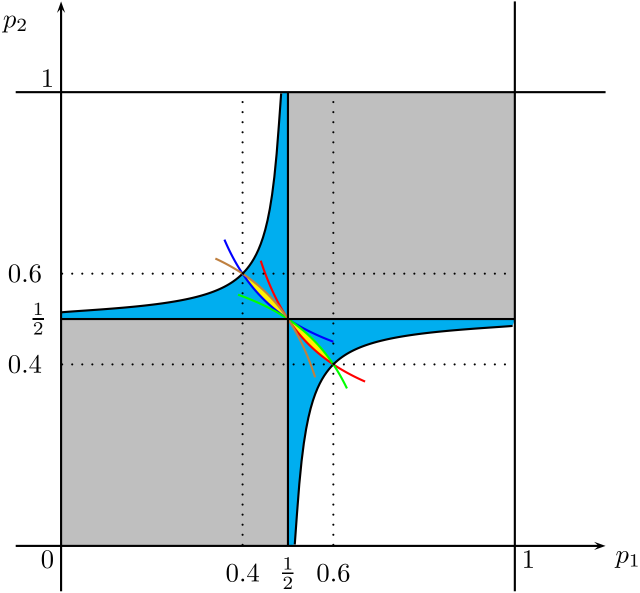
Region of Posteriors (p_1, p_2) for Channel Noise $\varepsilon = 0.35$



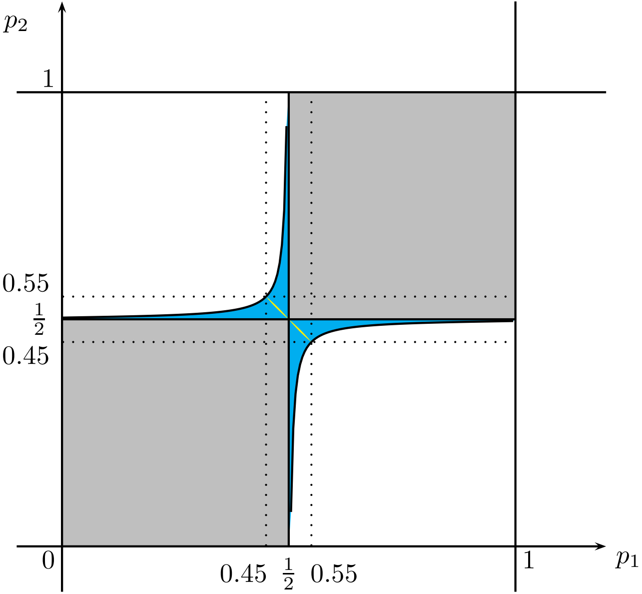
Region of Posteriors (p_1, p_2) for Channel Noise $\varepsilon = 0.375$



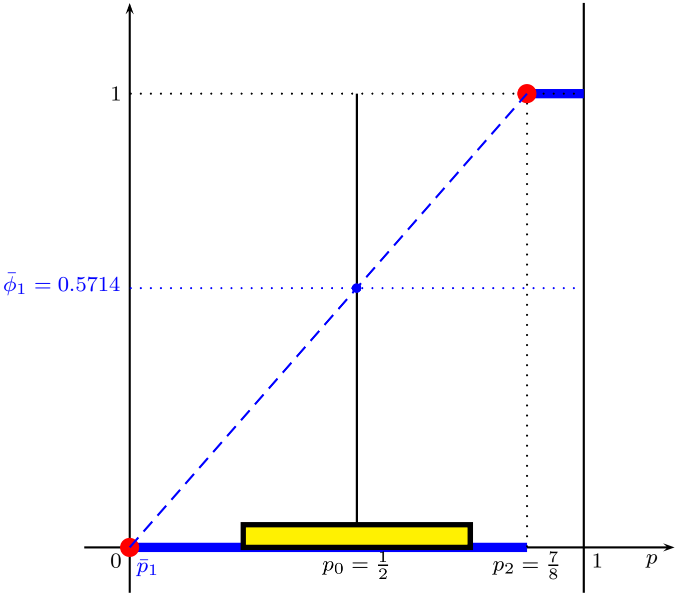
Region of Posteriors (p_1, p_2) for Channel Noise $\varepsilon = 0.40$



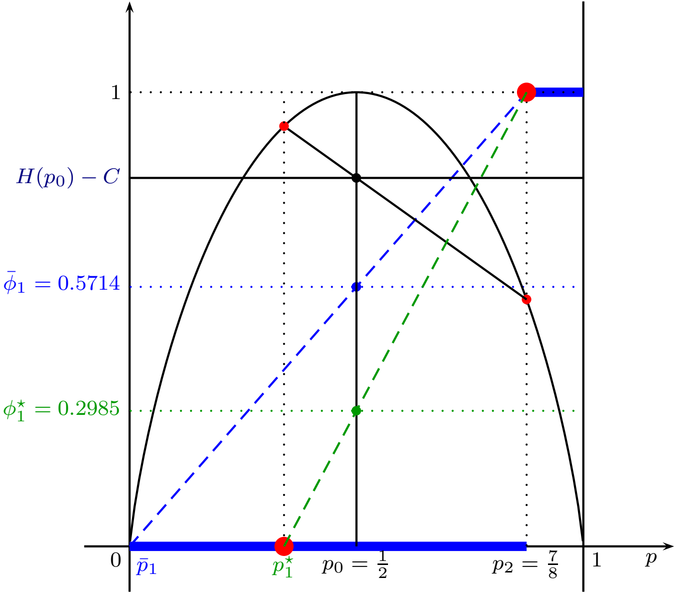
Region of Posteriors (p_1, p_2) for Channel Noise $\varepsilon = 0.45$



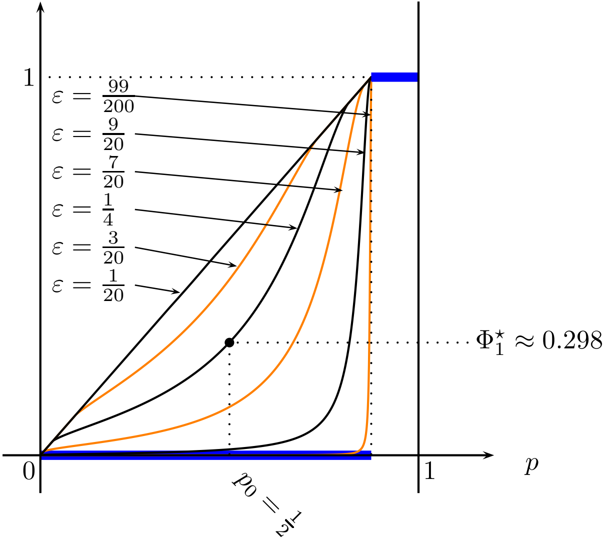
Example : One-sided investment for channel noise $\varepsilon = 0.25$



Example : One-sided investment for channel noise $\varepsilon = 0.25$



Optimal value for noise parameters ε



Sketch of proof

- Converse proof follows from identification of the auxiliary R.V. $W_T = (Y^n, T)$.

- Achievability proof

1) Sequences are jointly typical : Shannon 1948 "random coding scheme"

$(U^n, W^n) \in A(\mathcal{Q})$, for the target probability distribution $\mathbb{P}(u) \times \mathcal{Q}(w|u)$,

2) Control of the Posterior Beliefs induced by the coding process :

$$\mathbb{P}_\sigma(U_i|Y^n) \sim \mathcal{Q}(U_i|W_i)$$

$$\mathbb{E}_\sigma\left[\frac{1}{n}\sum_{i=1}^{n} D\Big(\mathbb{P}_\sigma(U_i|Y^n, E_\delta^0)\Big|\Big|\mathcal{Q}(U_i|W_i)\Big)\right] \leq \varepsilon$$

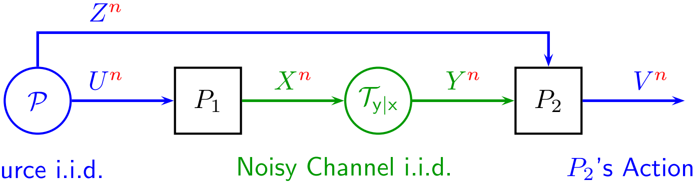
[Le Treust and Tomala, 2018]

"Information-Theoretic Limits of Strategic Communication"

https ://arxiv.org/abs/1807.05147

Joint Wyner-Ziv and channel codings

Utility Functions : $\Phi_k(\sigma,\tau) = \mathbb{E}_{\sigma,\tau}\left[\frac{1}{n}\sum_{i=1}^{n}\phi_k(u_i,v_i)\right]$ for player $k \in \{1,2\}$



Player P_1's Strategy :

$\sigma : \mathcal{U}^n \to \Delta(\mathcal{X}^n)$

Player P_2's Strategy :

$\tau : \mathcal{Y}^n \times \mathcal{Z}^n \to \mathcal{V}^n$

- Player P_1 chooses and announces strategy $\sigma(x^n|u^n)$ (commitment power).
- $(U^n, Z^n, X^n, Y^n) \sim \prod_{i=1}^{n}\mathcal{P}(u_i,z_i) \times \sigma(x^n|u^n) \times \prod_{i=1}^{n}\mathcal{T}(y_i|x_i) = \mathbb{P}_\sigma$
- Sequences (Y^n, Z^n) are observed by Player P_2.
- Player P_2 chooses a sequence of actions with $v^n = \tau(y^n, z^n)$.

Solution

Auxiliary random variable W with $|\mathcal{W}| = \min\left(|\mathcal{U}|+1, |\mathcal{V}|^{|\mathcal{Z}|}\right)$

$$\mathbb{Q}_0 = \left\{ \mathcal{P}_{\mathsf{uz}}(u,z) \times \mathcal{Q}(w|u), \quad \text{s.t.,} \quad \max_{\mathcal{P}(x)} I(X;Y) - I(U;W|Z) \geq 0 \right\}$$

$$\mathbb{Q}_2\big(\mathcal{Q}(u,z,w)\big) = \text{argmax}_{\mathcal{Q}(v|z,w)} \mathbb{E}_{\substack{\mathcal{Q}(u,z,w) \\ \times \mathcal{Q}(v|z,w)}} \Big[\phi_2(U,Z,V)\Big]$$

Define the optimal utility $\Phi_1^\star$ for Player P_1 :

$$\Phi_1^\star = \sup_{\mathcal{Q}(u,z,w) \in \mathbb{Q}_0} \min_{\substack{\mathcal{Q}(v|z,w) \in \\ \mathbb{Q}_2\left(\mathcal{Q}(u,z,w)\right)}} \mathbb{E}_{\substack{\mathcal{Q}(u,z,w) \\ \times \mathcal{Q}(v|z,w)}} \Big[\phi_1(U,Z,V)\Big]$$

Reformulation as an optimal splitting problem

Markov chain $W -\!\!\ominus\!\!- U -\!\!\ominus\!\!- Z \quad \Longleftrightarrow \quad \mathbb{P}(u,z,w) = \mathcal{P}(u,z)\mathbb{P}(w|u), \ \forall (u,z,w)$

$$\Longrightarrow \quad \mathbb{P}(u|z,w) = \frac{\mathbb{P}(u|w)\mathcal{P}(z|u)}{\sum_{u'} \mathbb{P}(u'|w)\mathcal{P}(z|u')}, \ \forall (u,z,w)$$

Encoder's utility reformulates as a function $\Psi_1(p)$ of decoder's belief $p(u)$:

$$\Psi_1(p) = \sum_{u,z} p(u) \cdot \mathcal{P}(z|u) \cdot \psi_1\left(\frac{p(u) \cdot \mathcal{P}(z|u)}{\sum_{u'} p(u') \cdot \mathcal{P}(z|u')}\right)$$

Conditional entropy $H(U|Z)$ reformulates as a function $h(p)$ of belief $p(u)$:

$$h(p) = \sum_{u,z} p(u) \cdot \mathcal{P}(z|u) \cdot \log_2 \frac{\sum_{u'} p(u') \cdot \mathcal{P}(z|u')}{p(u) \cdot \mathcal{P}(z|u)}$$

Reformulation as an Optimal Splitting

Concavification with information constraint

$$\begin{aligned} \Phi_1^\star \quad &= \quad \sup \sum_w \lambda_w \cdot \Psi_1(p_w) \\ \text{s.t.} \quad & \quad \sum_w \lambda_w \cdot p_w(u) = \mathcal{P}(u) \\ \text{and} \quad & \quad \sum_w \lambda_w \cdot h(p_w) \geq H(U|Z) - C \end{aligned}$$

- Information constraint : $H(U|W,Z) \geq H(U|Z) - C \Longleftrightarrow I(U;W|Z) \leq C$
- Auxiliary RV W is the index of the posterior beliefs
- Problem's dimension is $|\mathcal{U}|$, Caratheodory implies : $|\mathcal{W}| = |\mathcal{U}| + 1$.

Main Result

- Player 2's Best Replies $BR_2(\sigma) = \arg\max_{\tau} \Phi_2(\sigma, \tau)$:

$$BR_2(\sigma) = \arg \max_{v^n = \tau(y^n, z^n)} \mathbb{E}_{\sigma,\tau}\left[\frac{1}{n}\sum_{i}^{n} \phi_2(u_i, v_i)\right]$$

Theorem MLT and Tomala 2018

We characterize the best payoff player P_1 can secure :

$$1) \qquad \forall n \in \mathbb{N},\ \forall \sigma, \qquad \min_{\tau \in BR_2(\sigma)} \Phi_1(\sigma, \tau) \leq \Phi_1^{\star},$$

$$2) \qquad \forall \varepsilon > 0,\ \exists \bar{n},\ \forall n \geq \bar{n},\ \exists \sigma, \qquad \min_{\tau \in BR_2(\sigma)} \Phi_1(\sigma, \tau) \geq \Phi_1^{\star} - \varepsilon.$$

Comments

Lemma MLT and Tomala 2018

The optimal splitting for $\Phi_1^\star$ *has a number of posteriors* $|\mathcal{W}|$ *restricted to :*

$$|\mathcal{W}| = \min\Big(|\mathcal{U}| + 1, |\mathcal{V}|^{|\mathcal{Z}|}\Big).$$

1) Extend the mapping $\Psi_1(p)$ on the domain :

$$\mathcal{D} = \{(p,h) \in \Delta(\mathcal{U}) \times \mathbb{R} : 0 \leq h \leq H(U|Z)\}$$

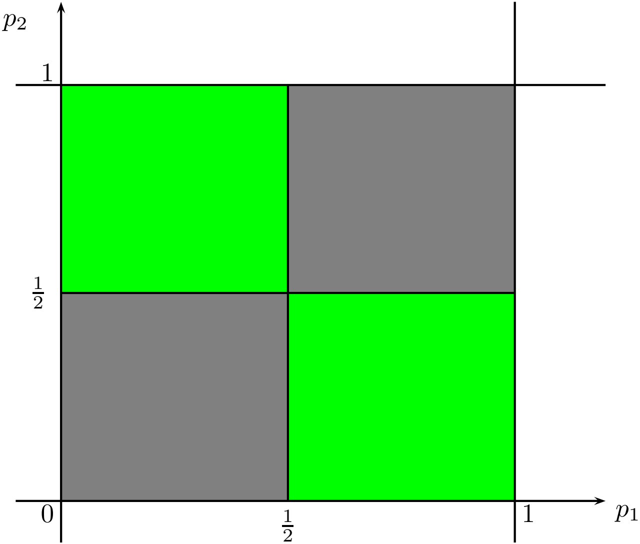
by $\Psi\big(p,h\big) = \Psi(p)$. Then, $\Phi_1^\star = \text{Cav}_{\,p,h}\,\Psi\Big(\mathcal{P}(u), H(U|Z) - C\Big)$

2) Lagrangian of the concavification :

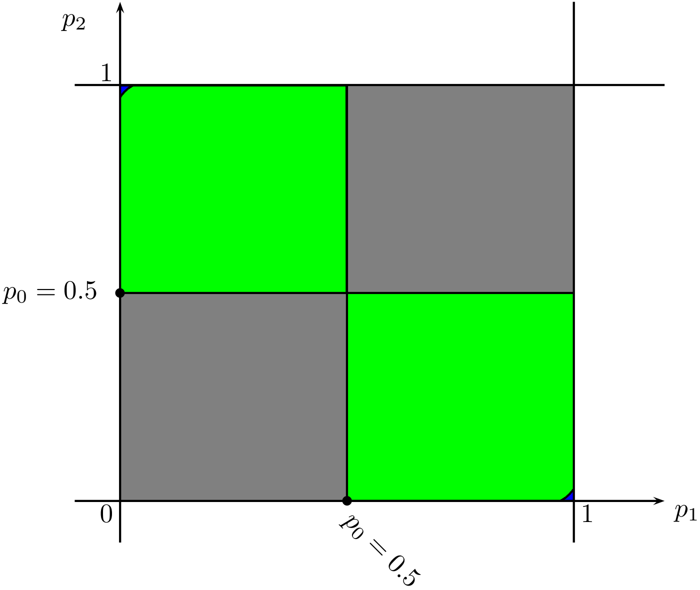
$$\Phi_1^\star = \inf_{t \geq 0}\Big\{\text{Cav}\,\big(\Psi + th\big)\big(\mathcal{P}(u)\big) - t\big(H(U|Z) - C\big)\Big\}$$

related with the **Cost of Information** in [Kamenica Gentzkow (AER) 2014]

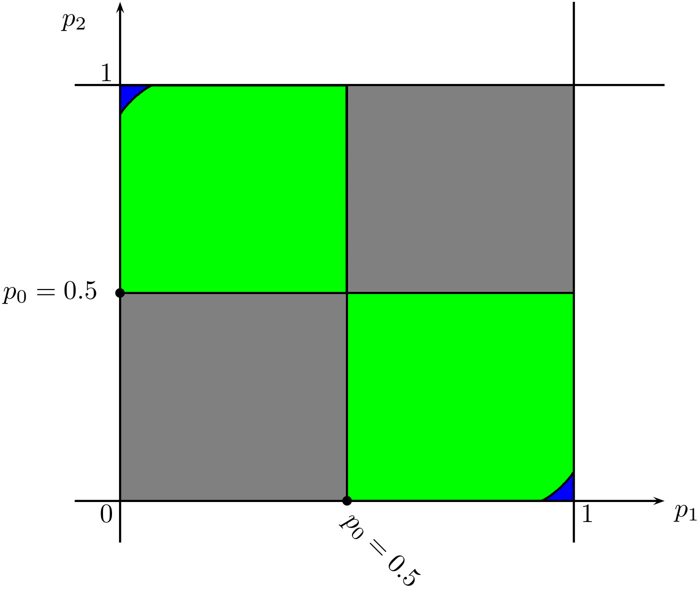
Region of Posteriors (p_1, p_2) for Capacity $C = 1$



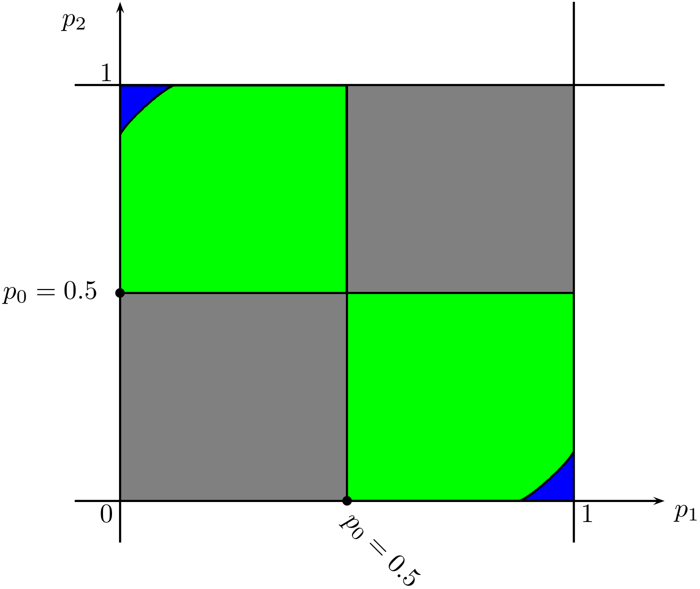
Region of Posteriors (p_1, p_2) for Capacity $C = 0.9$



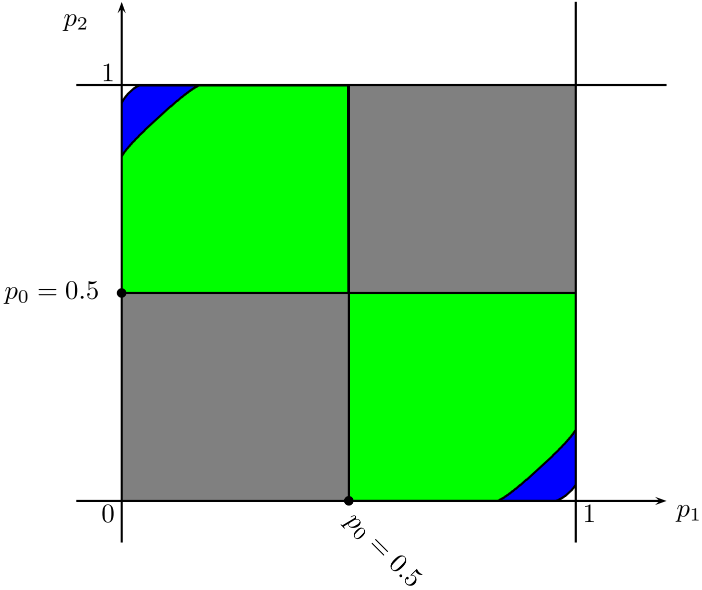
Region of Posteriors (p_1, p_2) for Capacity $C = 0.8$



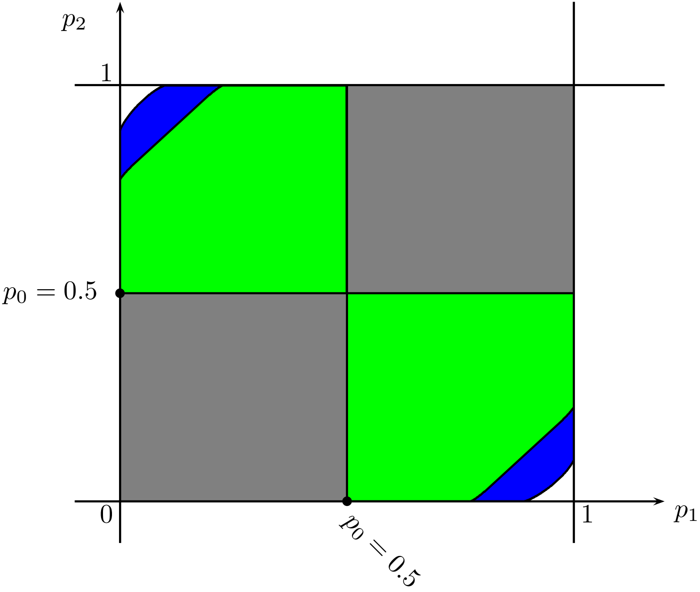
Region of Posteriors (p_1, p_2) for Capacity $C = 0.7$



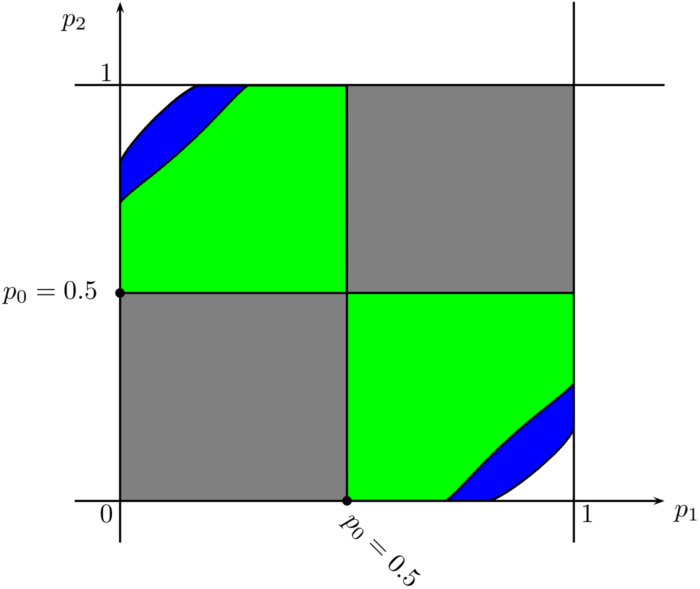
Region of Posteriors (p_1, p_2) for Capacity $C = 0.6$



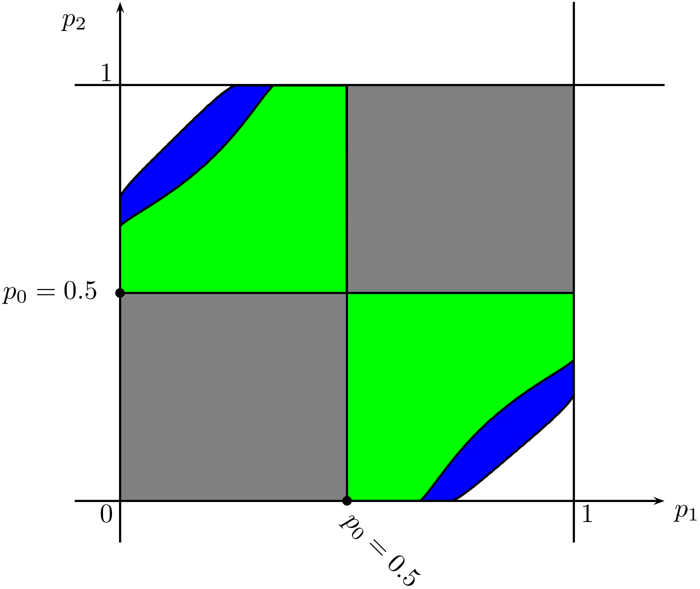
Region of Posteriors (p_1, p_2) for Capacity $C = 0.5$



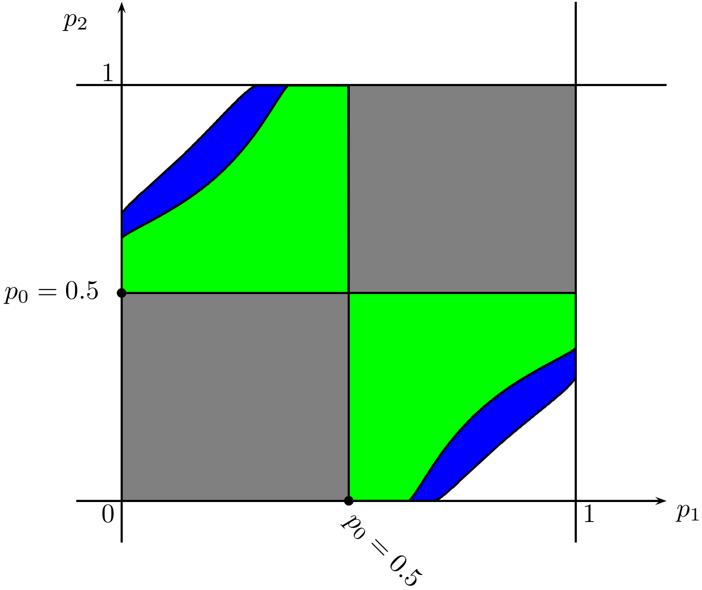
Region of Posteriors (p_1, p_2) for Capacity $C = 0.4$



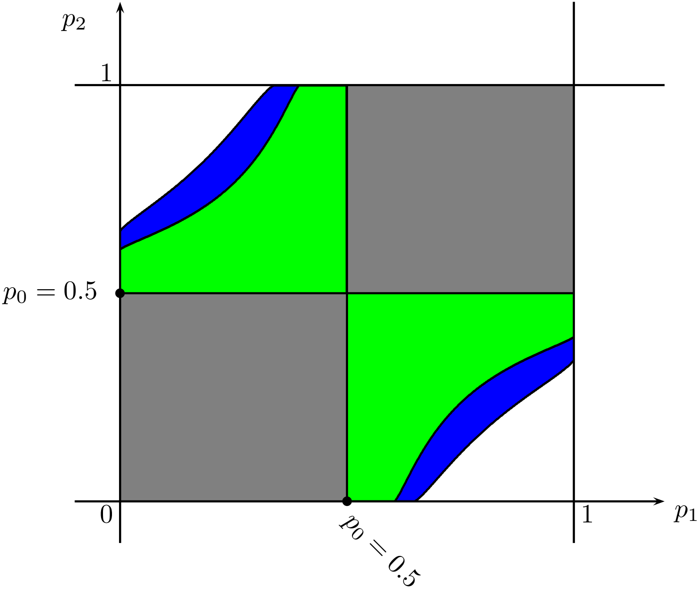
Region of Posteriors (p_1, p_2) for Capacity $C = 0.3$



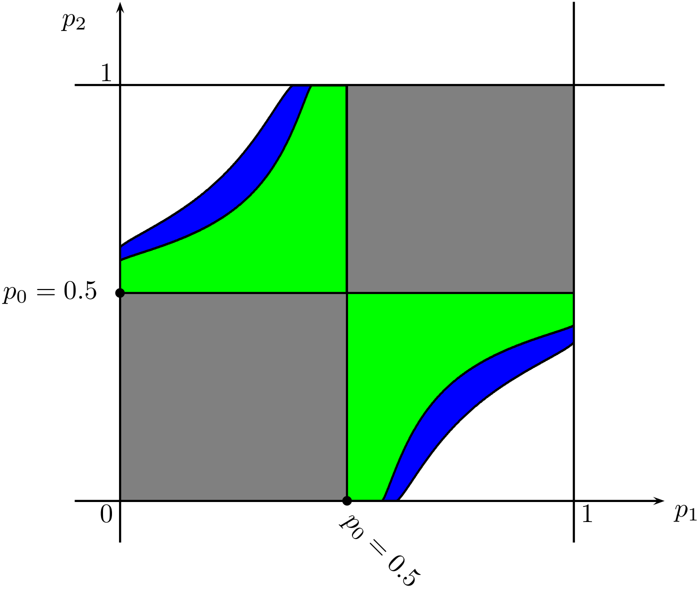
Region of Posteriors (p_1, p_2) for Capacity $C = 0.25$



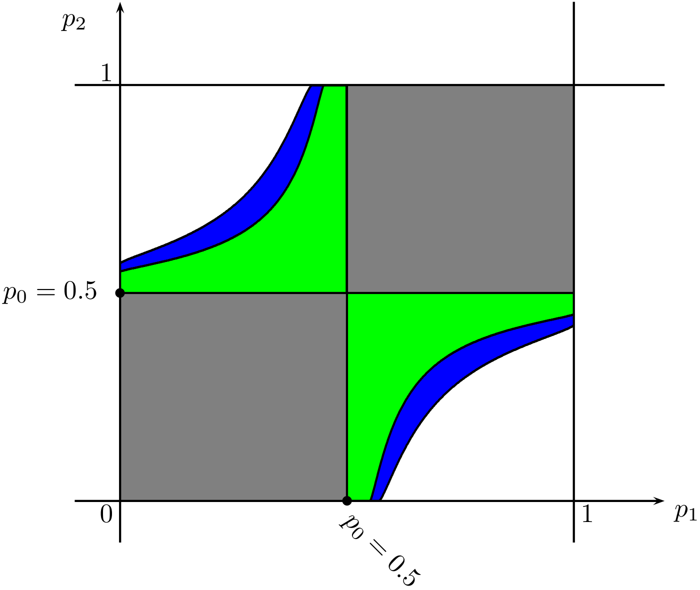
Region of Posteriors (p_1, p_2) for Capacity $C = 0.2$



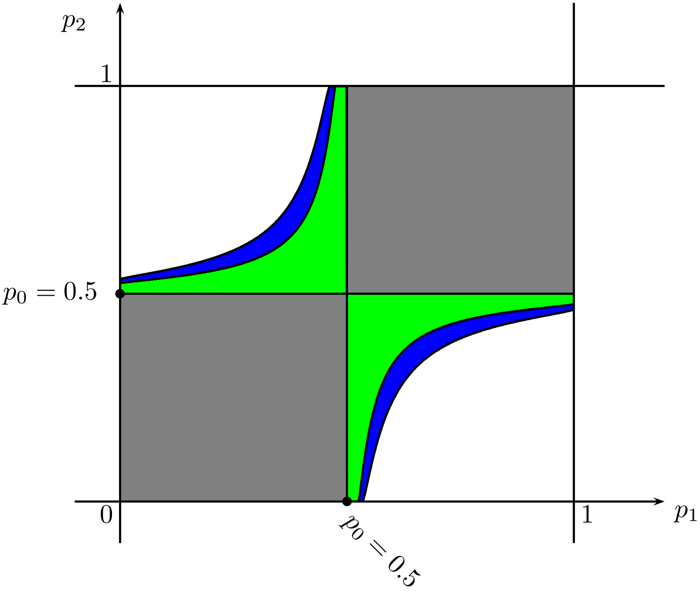
Region of Posteriors (p_1, p_2) for Capacity $C = 0.15$



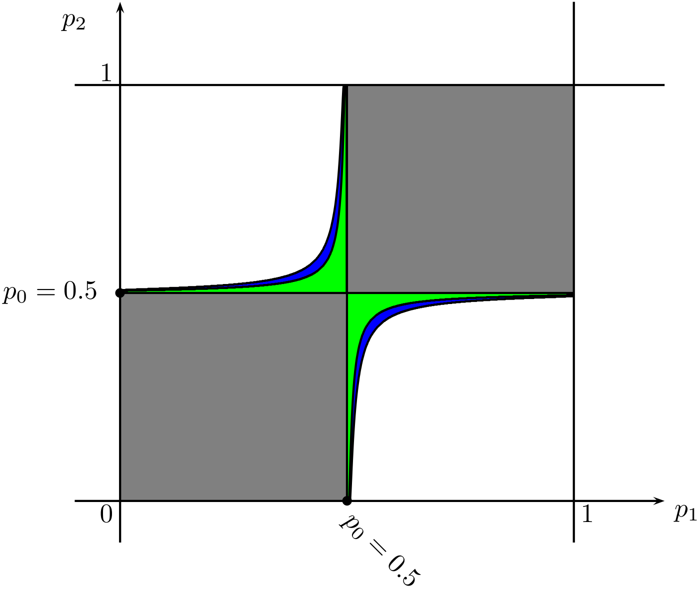
Region of Posteriors (p_1, p_2) for Capacity $C = 0.1$



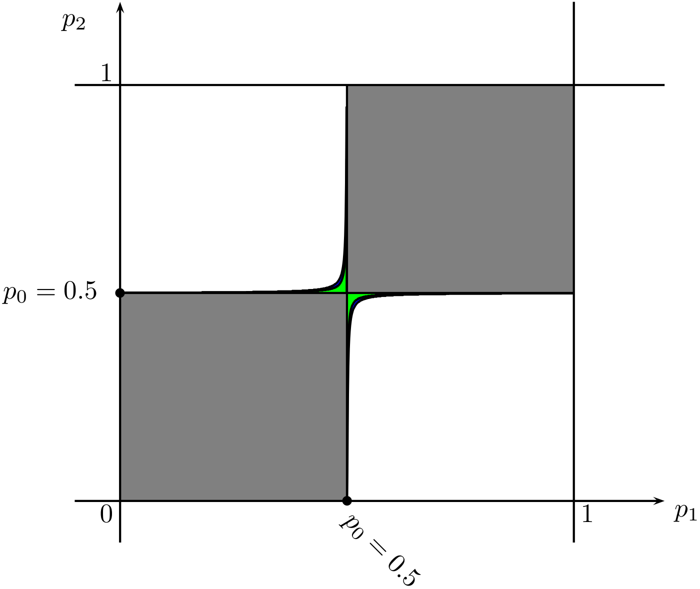
Region of Posteriors (p_1, p_2) for Capacity $C = 0.05$



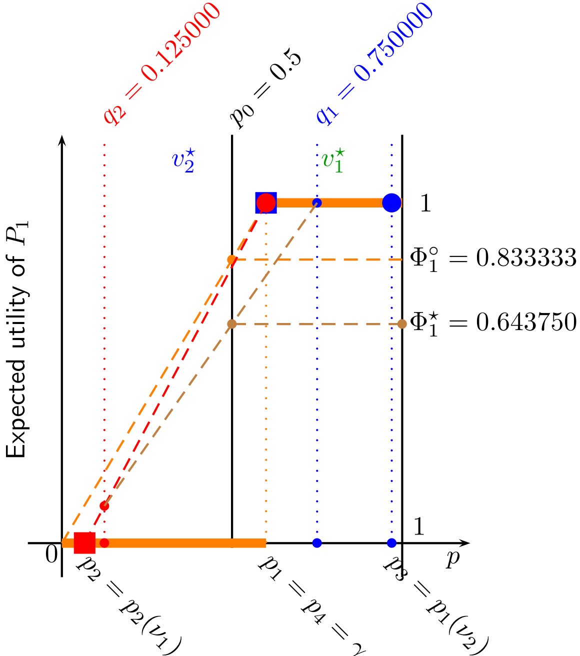
Region of Posteriors (p_1, p_2) for Capacity $C = 0.01$



Region of Posteriors (p_1, p_2) for Capacity $C = 0.001$



Example


$q_2 = 0.125000$
$p_0 = 0.5$
$q_1 = 0.750000$
$v_2^\star$
$v_1^\star$
1
$\Phi_1^\circ = 0.833333$
$\Phi_1^\star = 0.643750$
Expected utility of P_1
1
0
p
$p_2 = p_2(\nu_1)$
$p_1 = p_4 = \gamma$
$p_3 = p_1(\nu_2)$

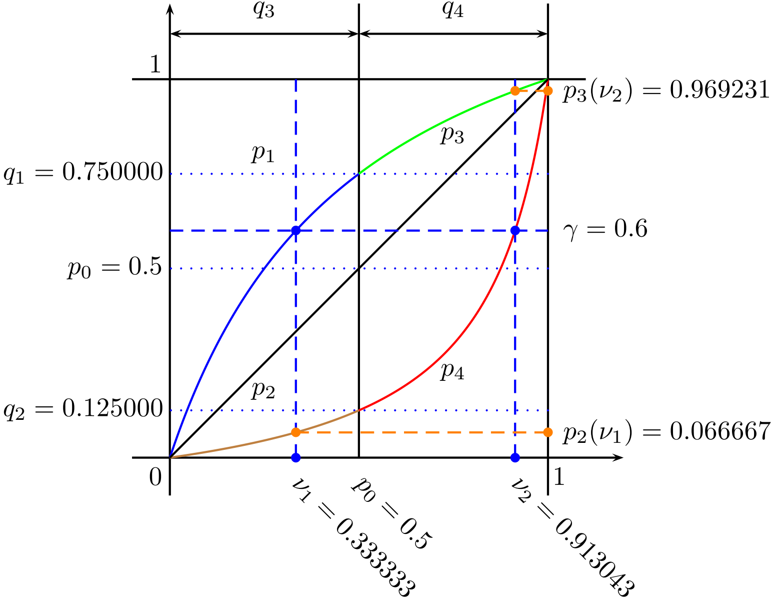
Example

We let $q_1 = \mathbb{P}(u_2|w_1)$, $q_2 = \mathbb{P}(u_2|w_2)$, $\delta_1 = \mathbb{P}(z_2|u_1)$, $\delta_2 = \mathbb{P}(z_1|u_2)$,

$$\begin{aligned}
p_1 = \mathcal{Q}(u_2|w_1, z_1) &= \frac{q_1 \cdot \delta_2}{(1-q_1)\cdot(1-\delta_1) + q_1 \cdot \delta_2}, \\
p_2 = \mathcal{Q}(u_2|w_1, z_2) &= \frac{q_1 \cdot (1-\delta_2)}{(1-q_1)\cdot\delta_1 + q_1 \cdot (1-\delta_2)}, \\
p_3 = \mathcal{Q}(u_2|w_2, z_1) &= \frac{q_2 \cdot \delta_2}{(1-q_2)\cdot(1-\delta_1) + q_2 \cdot \delta_2}, \\
p_4 = \mathcal{Q}(u_2|w_2, z_2) &= \frac{q_2 \cdot (1-\delta_2)}{(1-q_2)\cdot\delta_1 + q_2 \cdot (1-\delta_2)}.
\end{aligned}$$

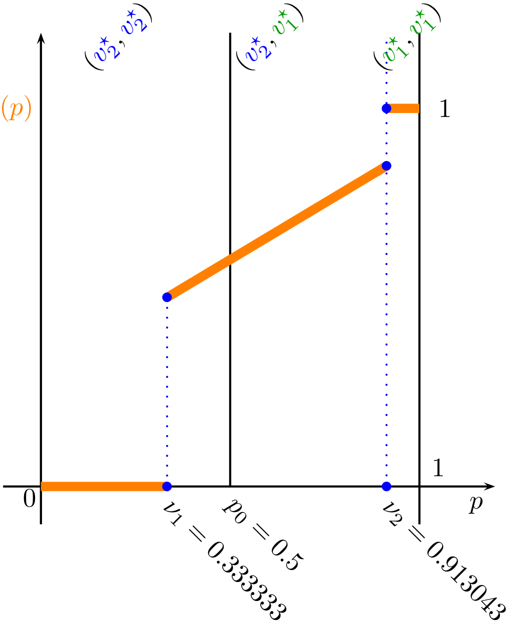
$$\begin{aligned}
p_1(q) &= \frac{q \cdot \delta_2}{(1-q)\cdot(1-\delta_1) + q \cdot \delta_2}, \\
p_2(q) &= \frac{q \cdot (1-\delta_2)}{(1-q)\cdot\delta_1 + q \cdot (1-\delta_2)}.
\end{aligned}$$

Example


q_3
q_4
1
$p_3(\nu_2) = 0.969231$
p_3
p_1
$q_1 = 0.750000$
$\gamma = 0.6$
$p_0 = 0.5$
p_4
p_2
$q_2 = 0.125000$
$p_2(\nu_1) = 0.066667$
0
1
$\nu_1 = 0.333333$
$p_0 = 0.5$
$\nu_2 = 0.913043$

Example

Encoder's Utility: $\Psi_1(p)$

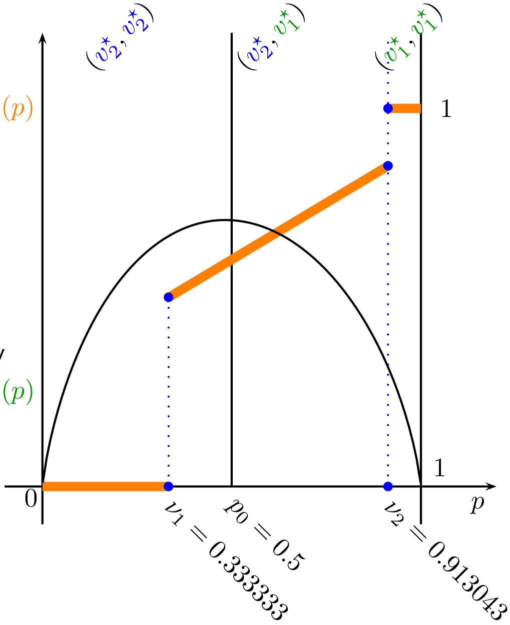


Example

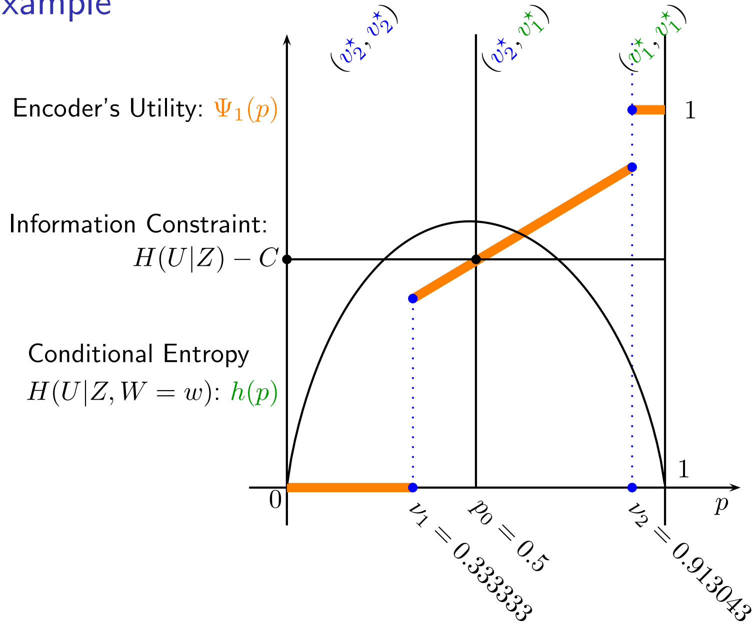
Encoder's Utility: $\Psi_1(p)$

Conditional Entropy

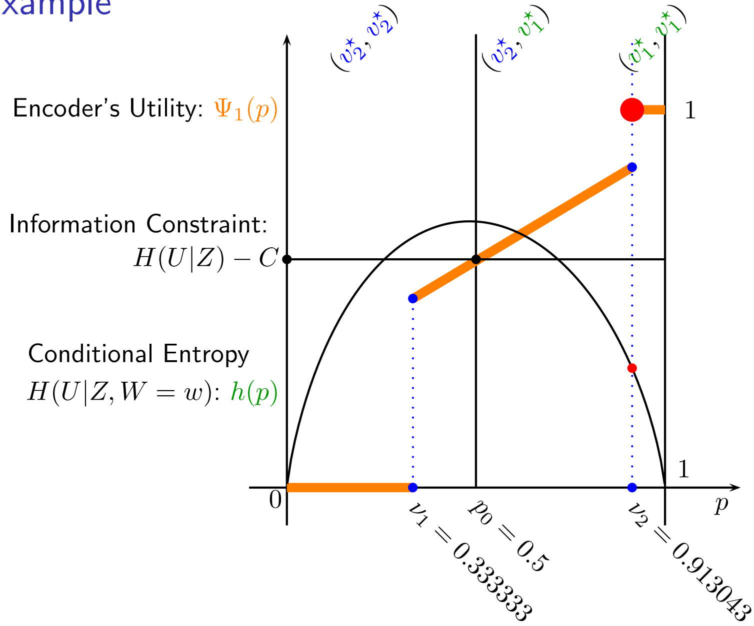
$H(U|Z, W = w)$: $h(p)$



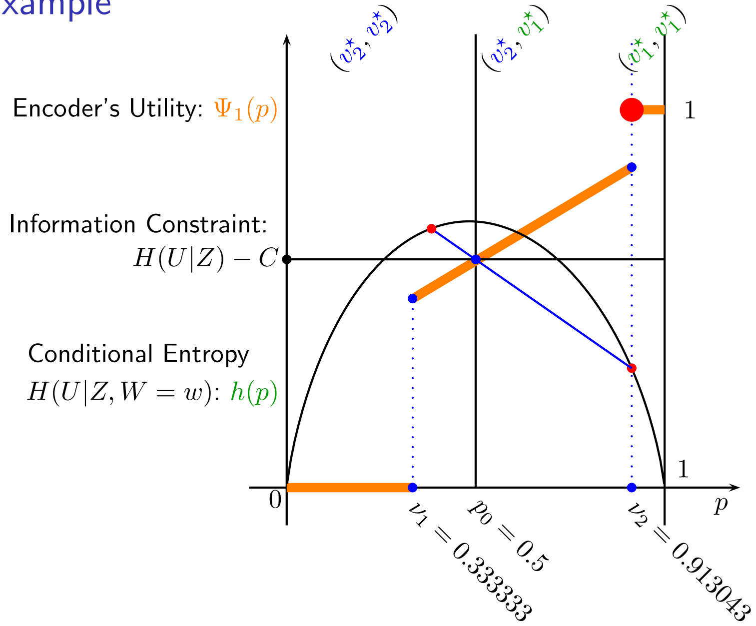
Example


$(v_2^\star, v_2^\star)$
$(v_2^\star, v_1^\star)$
$(v_1^\star, v_1^\star)$
Encoder's Utility: $\Psi_1(p)$
1
Information Constraint:
$H(U|Z) - C$
Conditional Entropy
$H(U|Z, W = w)$: $h(p)$
1
0
p
$\nu_1 = 0.333333$
$p_0 = 0.5$
$\nu_2 = 0.913043$

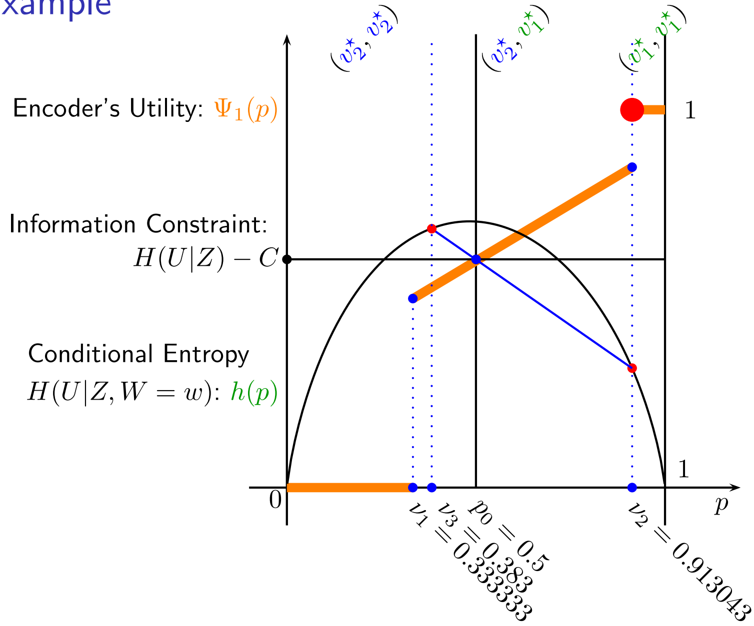
Example


$(\upsilon_2^\star, \upsilon_2^\star)$
$(\upsilon_2^\star, \upsilon_1^\star)$
$(\upsilon_1^\star, \upsilon_1^\star)$
Encoder's Utility: $\Psi_1(p)$
1
Information Constraint:
$H(U|Z) - C$
Conditional Entropy
$H(U|Z, W = w)$: $h(p)$
1
0
p
$\nu_1 = 0.333333$
$p_0 = 0.5$
$\nu_2 = 0.913043$

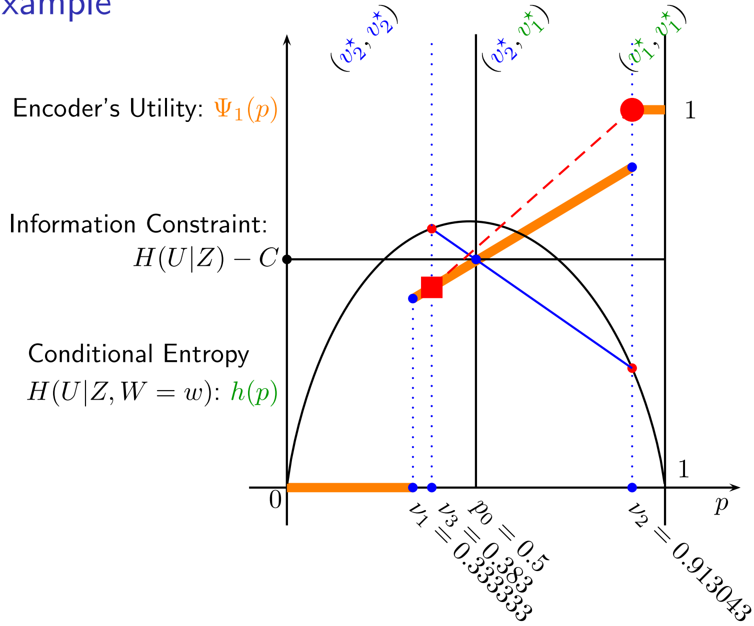
Example


$(v_2^\star, v_2^\star)$
$(v_2^\star, v_1^\star)$
$(v_1^\star, v_1^\star)$
Encoder's Utility: $\Psi_1(p)$
1
Information Constraint:
$H(U|Z) - C$
Conditional Entropy
$H(U|Z, W = w)$: $h(p)$
1
0
p
$\nu_1 = 0.333333$
$p_0 = 0.5$
$\nu_2 = 0.913043$

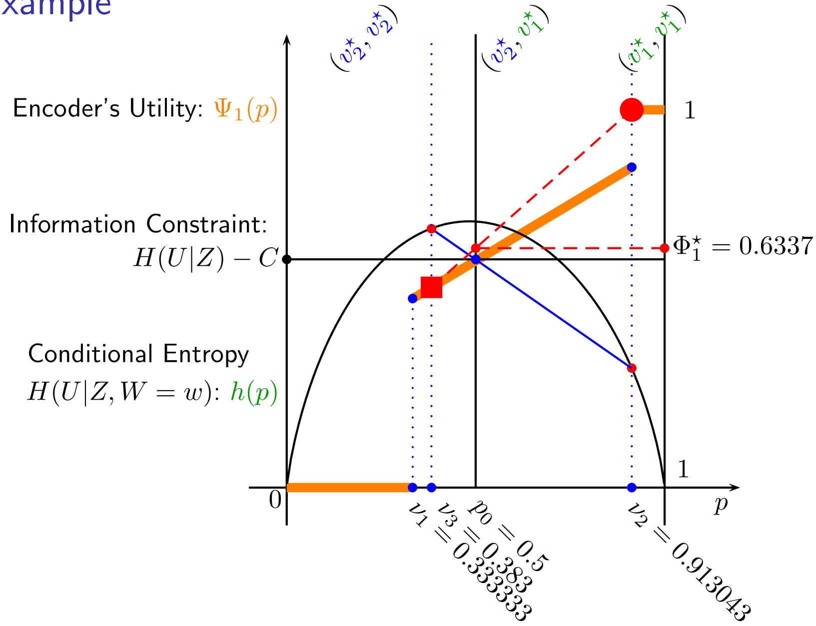
Example


Encoder's Utility: $\Psi_1(p)$
Information Constraint:
$H(U|Z) - C$
Conditional Entropy
$H(U|Z, W = w)$: $h(p)$
$(\nu_2^\star, \nu_2^\star)$
$(\nu_2^\star, \nu_1^\star)$
$(\nu_1^\star, \nu_1^\star)$
1
1
0
p
$\nu_1 = 0.333333$
$\nu_3 = 0.383$
$p_0 = 0.5$
$\nu_2 = 0.913043$

Example

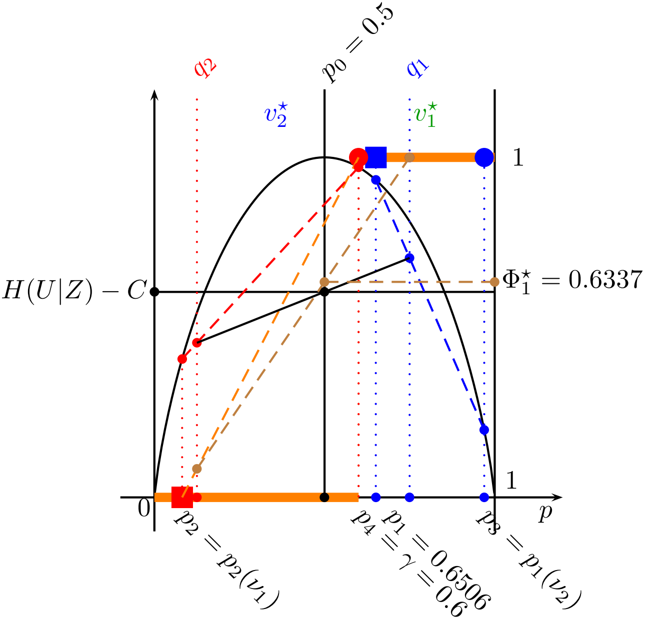

Encoder's Utility: $\Psi_1(p)$
Information Constraint:
$H(U|Z) - C$
Conditional Entropy
$H(U|Z, W = w)$: $h(p)$
$(v_2^\star, v_2^\star)$
$(v_2^\star, v_1^\star)$
$(v_1^\star, v_1^\star)$
1
1
0
p
$\nu_1 = 0.333333$
$\nu_3 = 0.383$
$p_0 = 0.5$
$\nu_2 = 0.913043$

Example



$(\nu_2^\star, \nu_2^\star)$
$(\nu_2^\star, \nu_1^\star)$
$(\nu_1^\star, \nu_1^\star)$
Encoder's Utility: $\Psi_1(p)$
1
Information Constraint:
$H(U|Z) - C$
$\Phi_1^\star = 0.6337$
Conditional Entropy
$H(U|Z, W = w)$: $h(p)$
1
0
p
$\nu_1 = 0.333333$
$\nu_3 = 0.383$
$p_0 = 0.5$
$\nu_2 = 0.913043$

Example


$p_0 = 0.5$
q_2
q_1
$v_2^\star$
$v_1^\star$
1
$H(U|Z) - C$
$\Phi_1^\star = 0.6337$
1
0
p
$p_2 = p_2(\nu_1)$
$p_4 = \gamma = 0.6$
$p_1 = 0.6506$
$p_3 = p_1(\nu_2)$

Achievability : Wyner-Ziv and Control of Posterior Beliefs

$$\mathbb{E}_\sigma\left[\frac{1}{n}\sum_{i=1}^{n} D\left(\mathcal{P}_\sigma(U_i|Y^n,Z^n,E_\delta^0)\middle\|\mathcal{Q}(U_i|W_i,Z_i)\right)\right]$$

$$= \frac{1}{n}\sum_{\substack{(u^n,z^n,\\ w^n,y^n)\in A_\delta}} \mathcal{P}_\sigma(u^n,z^n,w^n,y^n|E_\delta^0)\log_2\frac{1}{\prod_{i=1}^n \mathcal{Q}(u_i|w_i,z_i)} - \frac{1}{n}\sum_{i=1}^{n} H(U_i|Y^n,Z^n,E_\delta^0)$$

$$\leq H(U|W,Z)+\delta-\frac{1}{n}H(U^n|W^n,Y^n,Z^n,E_\delta^0)$$

$$\leq H(U|W,Z)-\frac{1}{n}H(U^n|W^n,Z^n,E_\delta^0)+\delta$$

$$\left[Z^n -\!\!\ominus\!\!- U^n -\!\!\ominus\!\!- W^n -\!\!\ominus\!\!- Y^n \Longrightarrow H(U^n|W^n,Z^n)=H(U^n|W^n,Y^n,Z^n)\right]$$

$$= H(U|W,Z)-\frac{1}{n}H(U^n|E_\delta^0)+\frac{1}{n}I(U^n;W^n|E_\delta^0)+\frac{1}{n}H(Z^n|W^nE_\delta^0)-\frac{1}{n}H(Z^n|U^nW^nE_\delta^0$$

$$= H(U|W,Z)-H(U)+I(U;W)+H(Z|W)-H(Z|U,W)+5\delta$$

$$= -I(U;W,Z)+I(U;W)+I(U;Z|W)+5\delta=5\delta$$

i.i.d. source, codebook size, typical sequences, i.i.d. source + $Z^n -\!\!\ominus\!\!- U^n -\!\!\ominus\!\!- W^n$

Sketch of Converse Proof :

Markov chain $Y^n -\!\!\!\!\circ\!\!\!\!- X^n -\!\!\!\!\circ\!\!\!\!- (U^n, Z^n)$ implies :

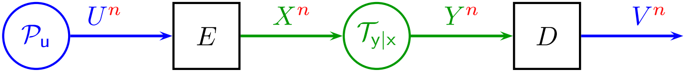
$$\begin{aligned}
0 &\leq I(X^n;Y^n) - I(U^n,Z^n;Y^n) \leq I(X^n;Y^n) - I(U^n;Y^n|Z^n) \\
&= H(Y^n) - H(Y^n|X^n) - H(U^n|Z^n) + H(U^n|Y^n,Z^n) \\
&\leq \sum_{i=1}^{n} H(Y_i) - \sum_{i=1}^{n} H(Y_i|X_i) - \sum_{i=1}^{n} H(U_i|Z_i) + \sum_{i=1}^{n} H(U_i|Y^n, Z^{-i}, Z_i) \\
&= \sum_{i=1}^{n} I(X_i;Y_i) - \sum_{i=1}^{n} I(U_i;W_i|Z_i) \\
&\leq n \cdot \max_{\mathcal{P}(x)} I(X;Y) - \sum_{i=1}^{n} I(U_i;W_i|Z_i) \\
&= n \cdot \left(\max_{\mathcal{P}(x)} I(X;Y) - I(U;W_T,T|Z) \right) \\
&= n \cdot \left(\max_{\mathcal{P}(x)} I(X;Y) - I(U;W|Z) \right).
\end{aligned}$$

Identification $W = (Y^n, Z^{-T}, T)$ satisfies both Markov chains :

$$Z -\!\!\!\!\circ\!\!\!\!- U -\!\!\!\!\circ\!\!\!\!- W \text{ and } V -\!\!\!\!\circ\!\!\!\!- (Z,W) -\!\!\!\!\circ\!\!\!\!- U.$$

Mismatched distortion functions

Joint source-channel coding scheme

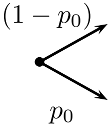


Source i.i.d.

Noisy Channel i.i.d.

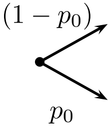
Mismatched distortion functions

$d_{\mathsf{e}}(u, v)$



	v_0	v_1
u_0	0	1
u_1	1	0

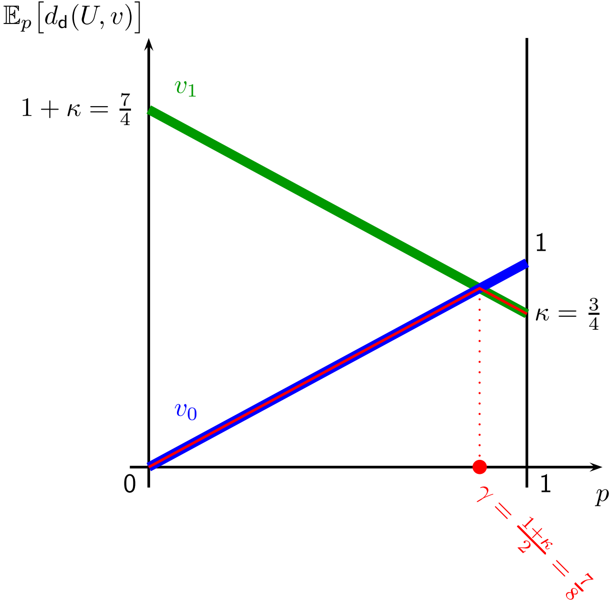
$d_{\mathsf{d}}(u, v)$



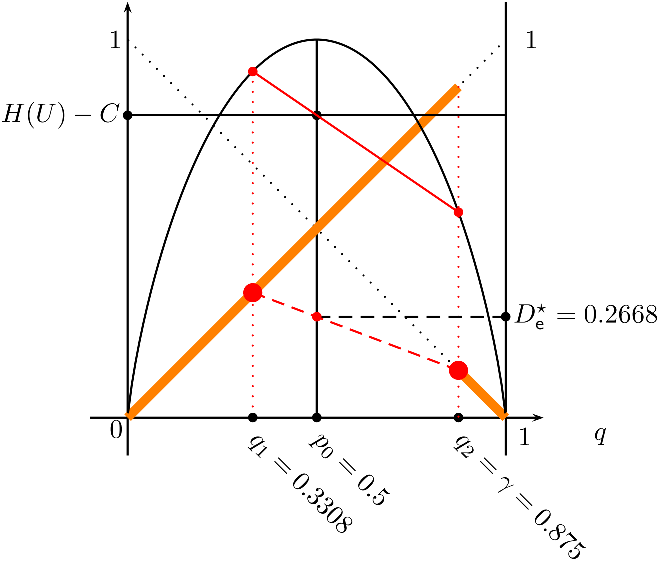
	v_0	v_1
u_0	0	$1 + \kappa$
u_1	1	κ

$\kappa \geq 0$ is an **extra cost**, e.g. energy, computing, symbol preference

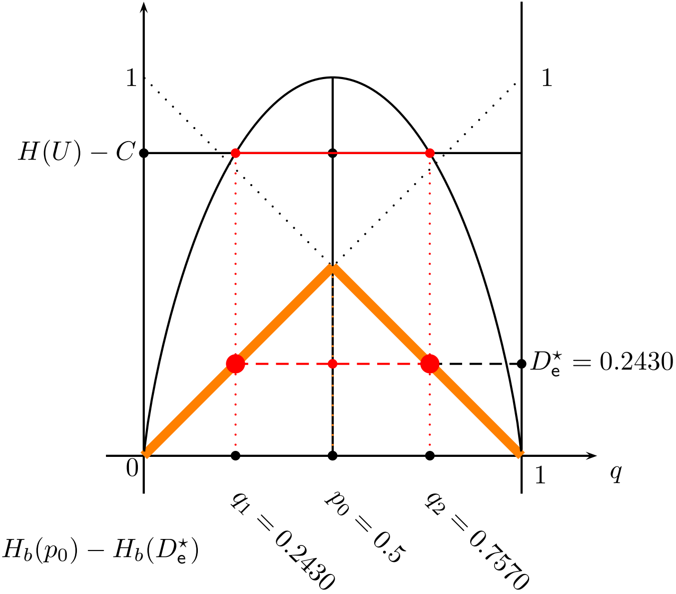
Decoder's best-reply symbol



Encoder's optimal distortion for $C = 0.2$ and $\kappa = \frac{3}{4}$



Shannon's rate-distortion function, $C = 0.2$ and $\kappa = 0$

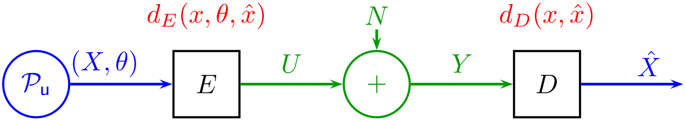


$C(D_{\mathsf{e}}^{\star}) = H_b(p_0) - H_b(D_{\mathsf{e}}^{\star})$

[Akyol Langbort Başar in Proc. IEEE 2017]

→ Information Theoretical view of Persuasion - Decentralized Stoch. Control

Distortions measures :



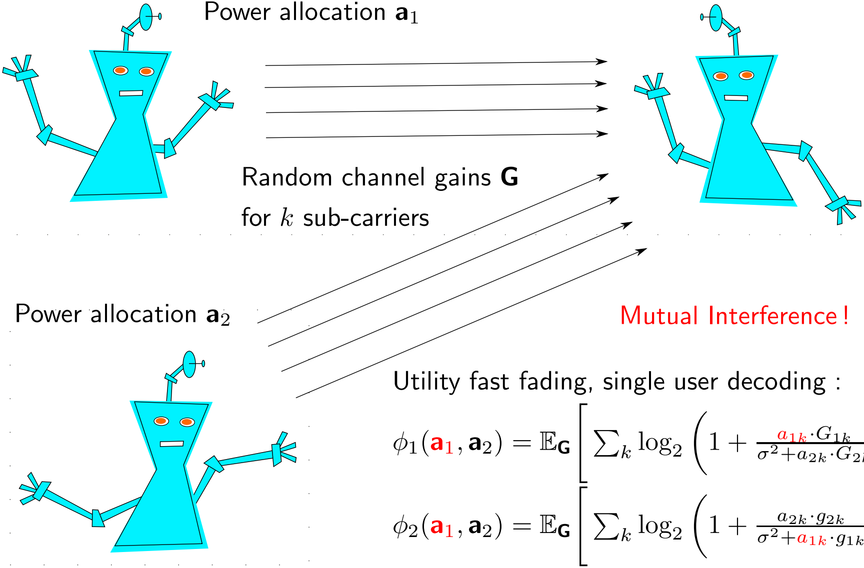
- **Gaussian** Source $(X,\theta) \sim \mathcal{N}(0, R_{X\theta})$ with $R_{X\theta} = \sigma_X^2 \begin{bmatrix} 1 & \rho \\ \rho & r \end{bmatrix}$, $\mathbb{E}[X^2] \le P_T$,
- **Quadratic** Distortions functions for encoder d_E and decoder d_D :

$$d_E(x,\theta,\hat{x}) = (x+\theta-\hat{x})^2, \qquad d_D(x,\hat{x}) = (x-\hat{x})^2$$

Theorem 7 [Akyol Langbort Başar in Proc. IEEE 2017]

$$\begin{aligned} U^\star &= \sqrt{\frac{P_T}{\sigma_X^2(1+2\alpha\rho+\alpha^2 r)}}(X+\alpha\theta), \\ \hat{X}^\star &= \mathbb{E}[X|Y], \qquad \text{with } \alpha = (-1+\sqrt{1+4(r+\rho)})/2(r+\rho) \end{aligned}$$

Example : Power Allocation Game for Parallel MAC



Power allocation $\mathbf{a}_1$

Random channel gains $\mathbf{G}$

for k sub-carriers

Power allocation $\mathbf{a}_2$

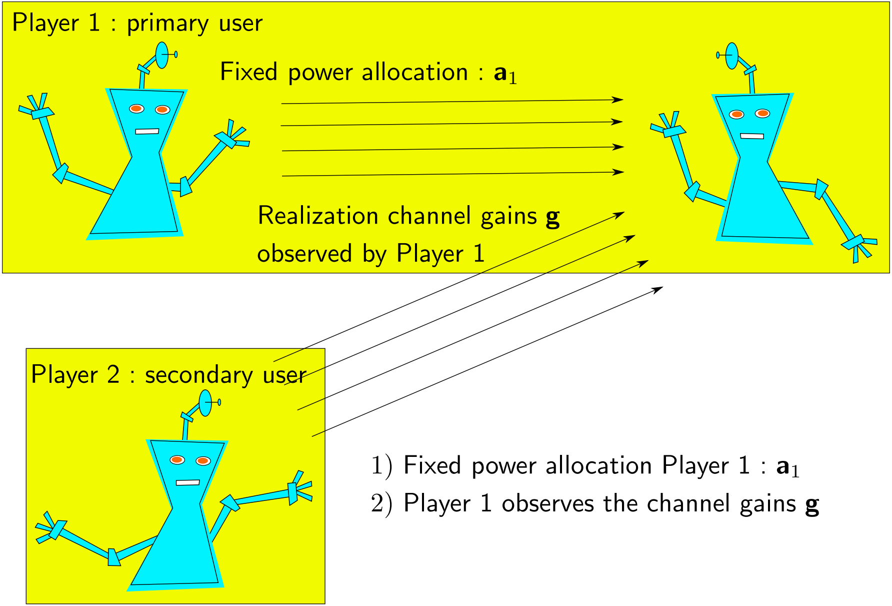
Mutual Interference !

Utility fast fading, single user decoding :

$$\phi_1(\mathbf{a}_1, \mathbf{a}_2) = \mathbb{E}_{\mathbf{G}}\left[\sum_k \log_2\left(1 + \frac{a_{1k} \cdot G_{1k}}{\sigma^2 + a_{2k} \cdot G_{2k}}\right)\right]$$

$$\phi_2(\mathbf{a}_1, \mathbf{a}_2) = \mathbb{E}_{\mathbf{G}}\left[\sum_k \log_2\left(1 + \frac{a_{2k} \cdot g_{2k}}{\sigma^2 + a_{1k} \cdot g_{1k}}\right)\right]$$

Strategic Transmission of Information



Player 1 : primary user

Fixed power allocation : $\mathbf{a}_1$

Realization channel gains $\mathbf{g}$

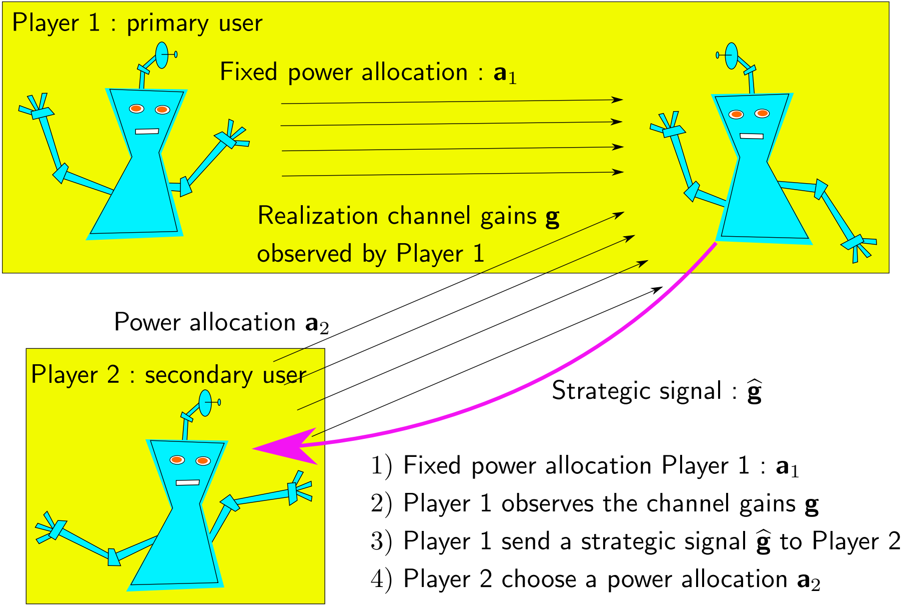
observed by Player 1

Player 2 : secondary user

$1)$ Fixed power allocation Player 1 : $\mathbf{a}_1$

$2)$ Player 1 observes the channel gains $\mathbf{g}$

Strategic Signaling of Channel Gains



Player 1 : primary user

Fixed power allocation : $\mathbf{a}_1$

Realization channel gains $\mathbf{g}$

observed by Player 1

Power allocation $\mathbf{a}_2$

Player 2 : secondary user

Strategic signal : $\widehat{\mathbf{g}}$

$1)$ Fixed power allocation Player 1 : $\mathbf{a}_1$

$2)$ Player 1 observes the channel gains $\mathbf{g}$

$3)$ Player 1 send a strategic signal $\widehat{\mathbf{g}}$ to Player 2

$4)$ Player 2 choose a power allocation $\mathbf{a}_2$

Example with Two Configurations

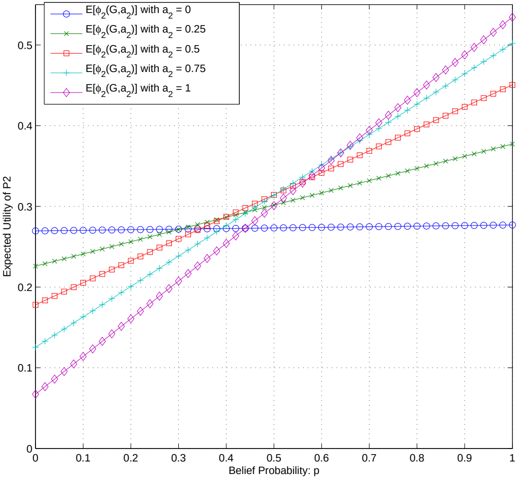
Power allocation game
with **two** parallel MACs :

- Belief proba. : $p \in [0, 1]$
- Channel gains :

	u_0	u_1
g_{11}	1.1878	0.1811
g_{12}	1.1566	1.4475
g_{21}	0.8407	0.0717
g_{22}	0.6293	0.6858

- Allocations : $a_1 = 0.16$ fixed
- $v \in \{0, 0.25, 0.5, 0.75, 1\}$

$$\phi_2(u, v) = \log_2\left(1 + \frac{v \cdot g_{21}}{\sigma^2 + a_1 \cdot g_{11}}\right) + \log_2\left(1 + \frac{(1 - v) \cdot g_{22}}{\sigma^2 + (1 - a_1) \cdot g_{12}}\right).$$



Player P_2's Best-Reply to her Belief

Power allocation game
with **two** parallel MACs :

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- Channel gains :

	u_0	u_1
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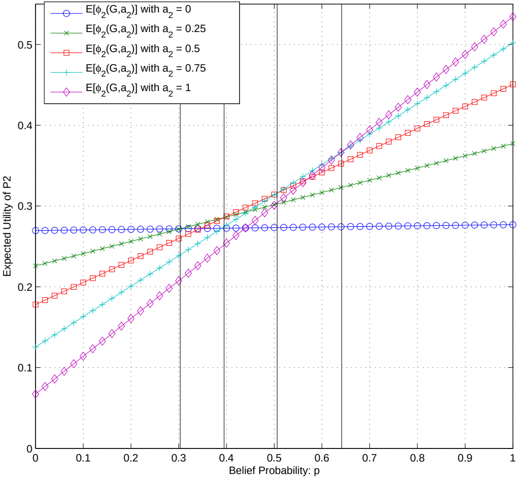

Expected Utility : $\mathbb{E}\left[\phi_2(U, v)\right] \;=\; p \cdot \phi_2(u_0, v) + (1 - p) \cdot \phi_2(u_1, v2)$

Player P_2's Best-Reply to her Belief

Power allocation game
with **two** parallel MACs :

- Belief proba. : $p \in [0, 1]$
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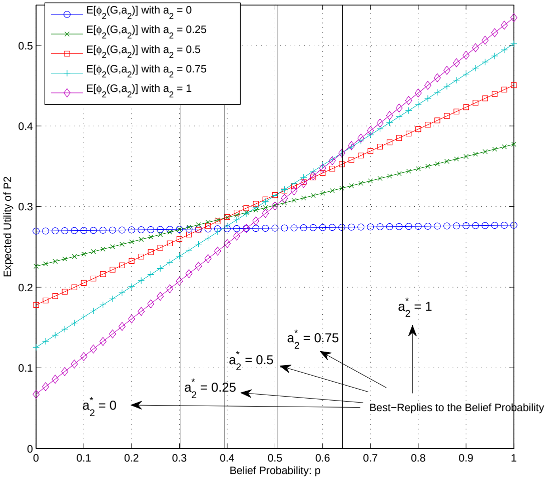
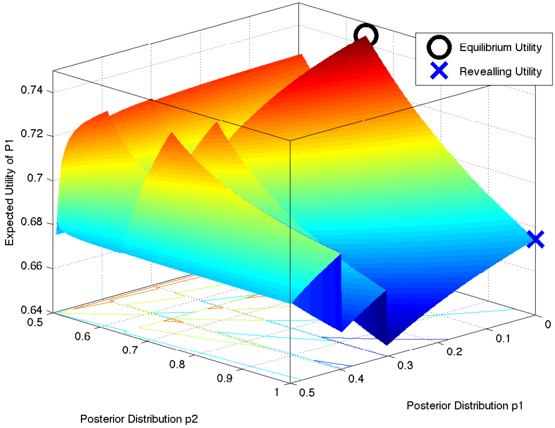


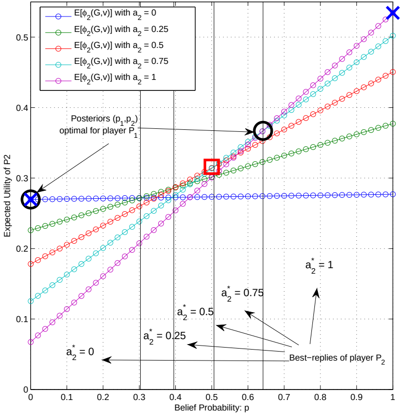
Expected Utility : $\mathbb{E}\left[\phi_2(U, v)\right] = p \cdot \phi_2(u_0, v) + (1 - p) \cdot \phi_2(u_1, v2)$

Expected Utility of Player P_1 for Posteriors (p_1, p_2)

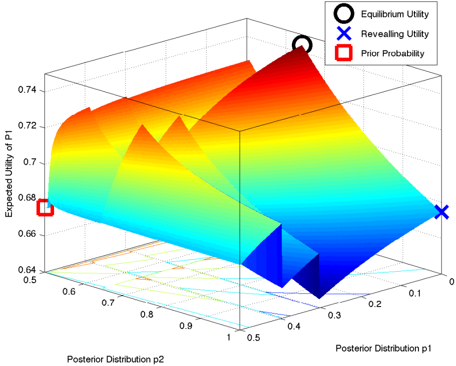


$$\mathbb{E}\left[\phi_1(U,v)\right] = \lambda\cdot\Big(p_1\cdot\phi_1(g_A,v^\star)+(1-p_1)\cdot\phi_1(g_B,v^\star)\Big)$$
$$+\quad (1-\lambda)\cdot\Big(p_2\cdot\phi_1(g_A,v^\star)+(1-p_2)\cdot\phi_1(g_B,v^\star)\Big)$$

Equilibrium Solution without Channel Noise



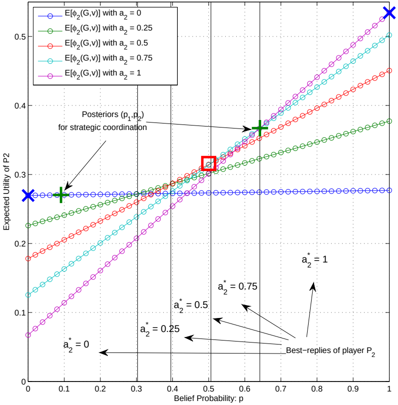
Utility of P_2

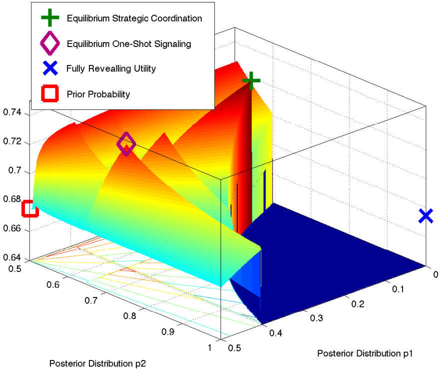


Utility of P_1

Equilibrium parameters : $(p_1, p_2) = (0, 0.6415), \quad (\alpha, \beta) = (1, 0.4424)$

Equilibrium Solution for Noisy Channel





Utility of P_2 Utility of P_1

$$\varepsilon = 0.25, \qquad (p_1, p_2) = (0.0910, 0.6420), \qquad (\alpha, \beta) = (0.9531, 0.4685)$$

Thank you !

https ://sites.google.com/site/maelletreust/

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