February 3, 2021

"RELAX" WORKSHOP ON GAMES

ON GAME EQUIVALENCES: ALGEBRAIC AND LOGICAL PERSPECTIVES

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WHEN ARE TWO GAMES THE SAME ?

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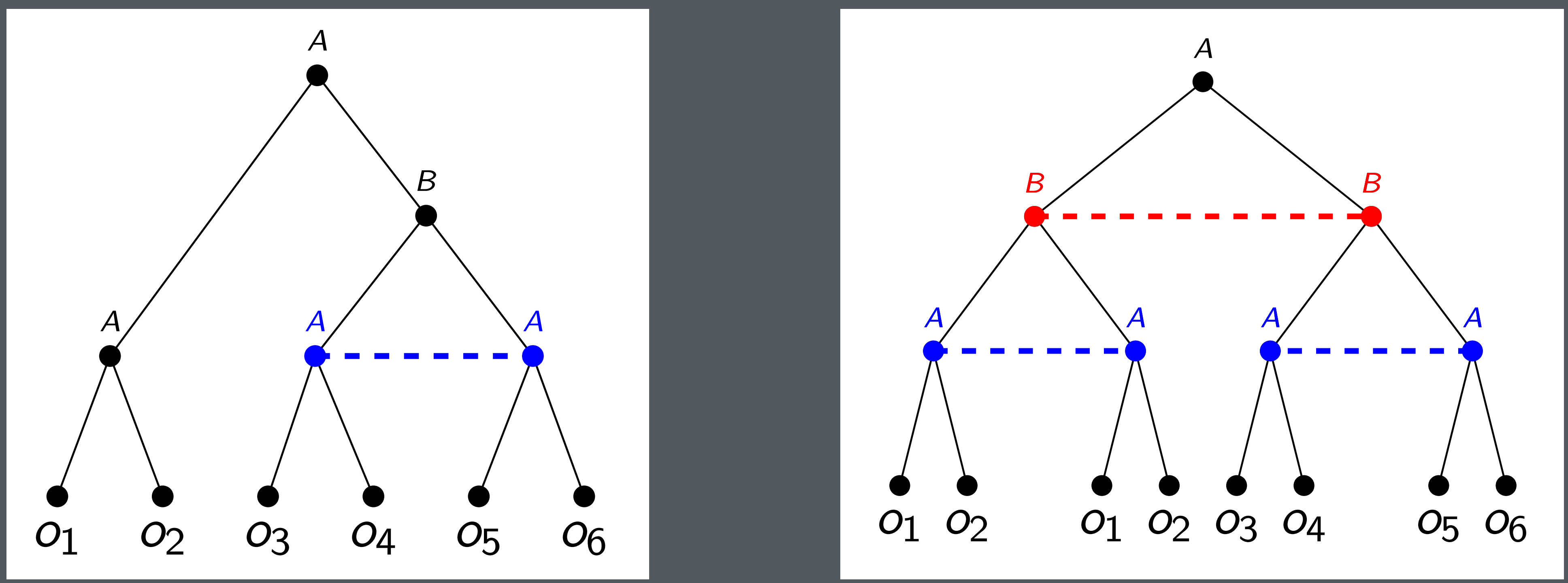
- From whose point of view ? (players, modelers)
- Different perspectives: transformations, structural, agents

Main focus: A high-level abstract framework of game forms

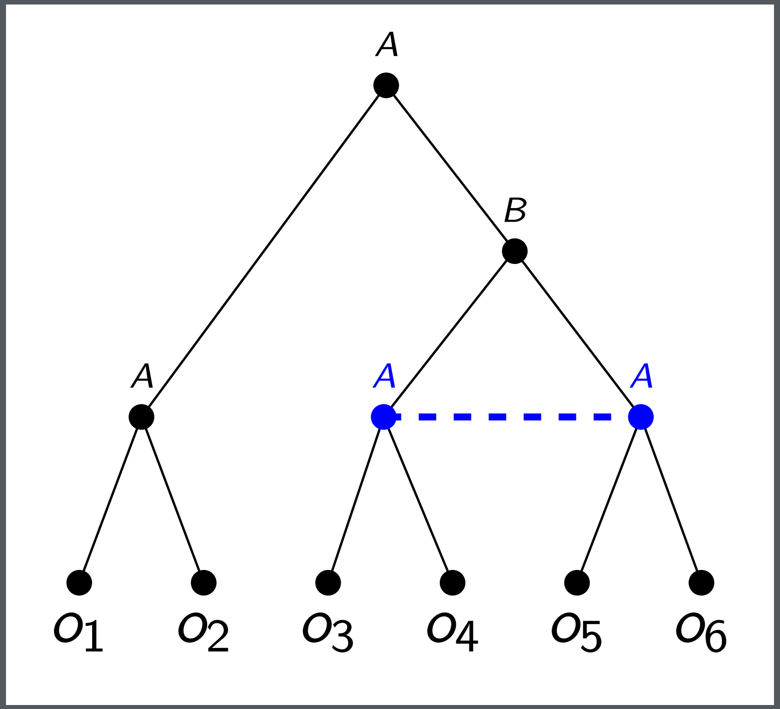
THOMPSON TRANSFORMATIONS

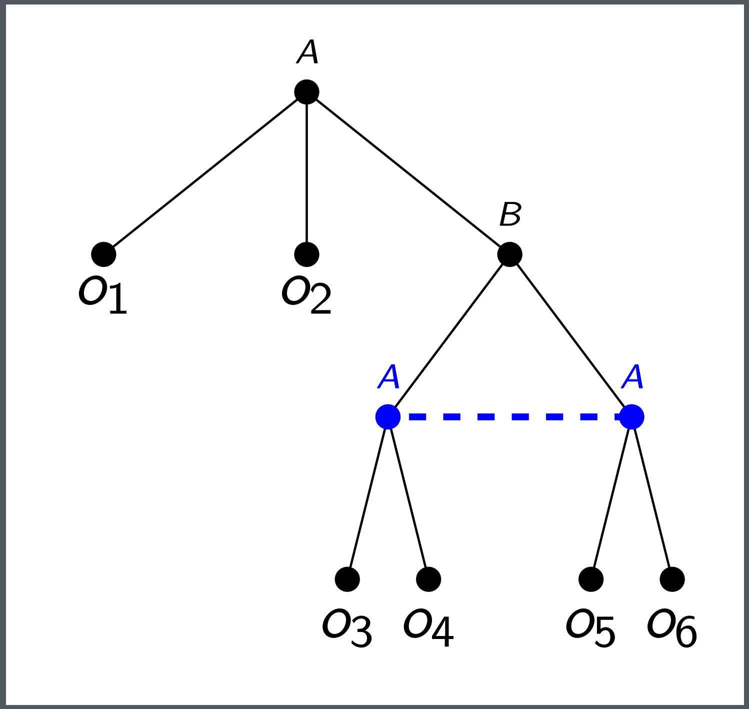
Game-theoretic analyses should not depend on “irrelevant” features of the mathematical description of the game

F.B. Thompson (1952)

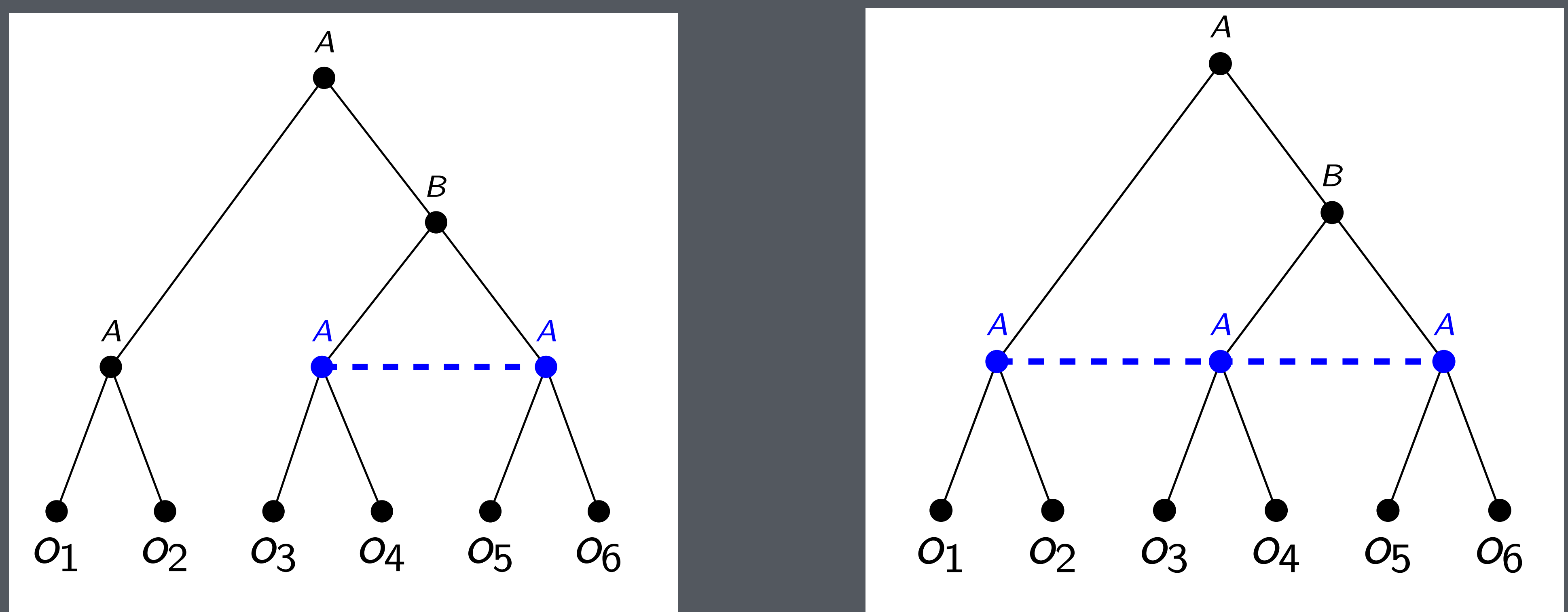


Addition of superfluous moves

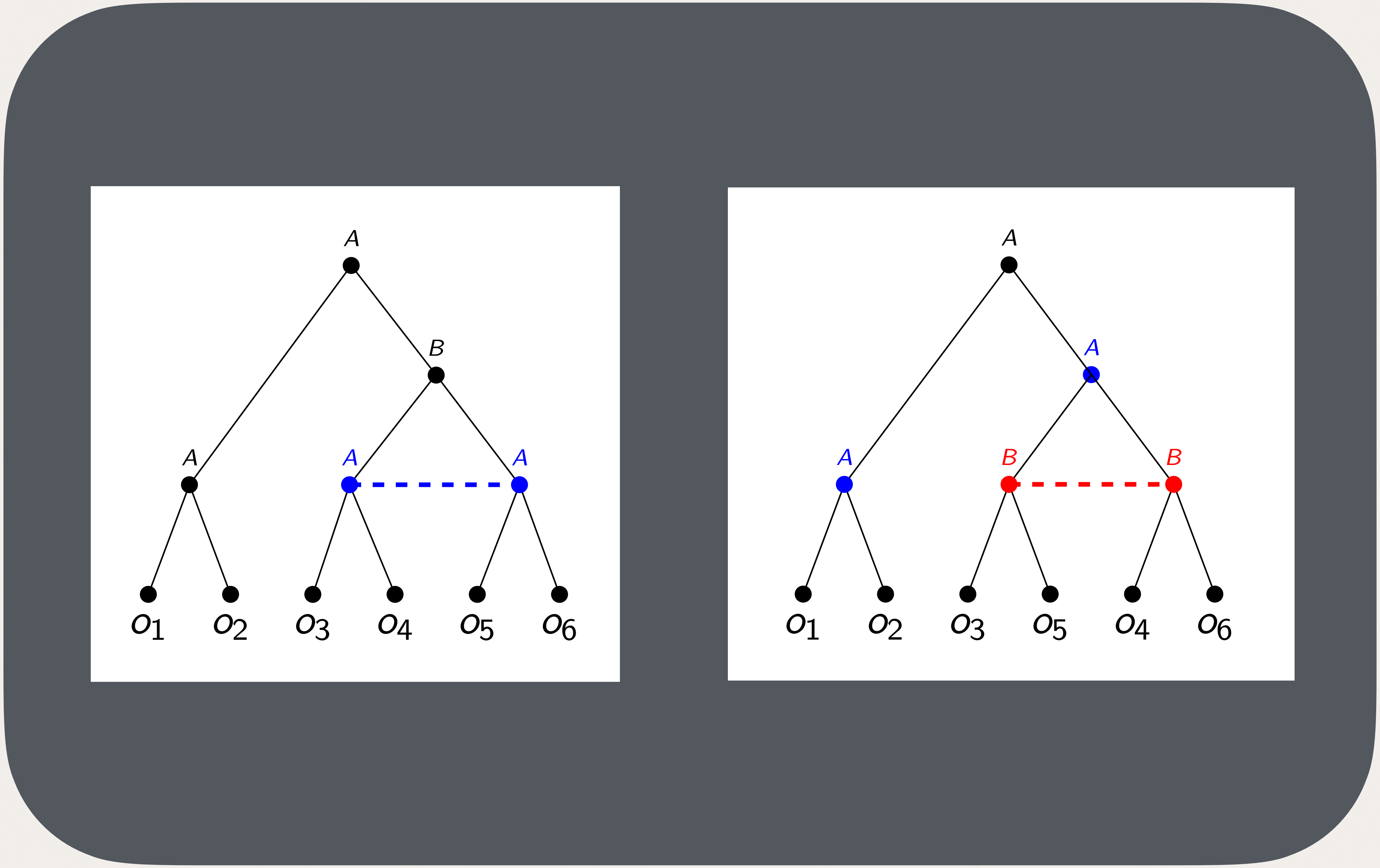




Coalescing of moves



Inflation/deflation



Interchange of moves

THOMPSON'S THEOREM

Each of the previous transformations preserve the reduced strategic form of the game. In any finite extensive form games of imperfect information, if any two games have the same reduced normal form then one can be obtained from the other by a sequence.

F.B. Thompson (1952)

OTHER EXAMPLES

S. Elmes and P.J. Reny (1994)

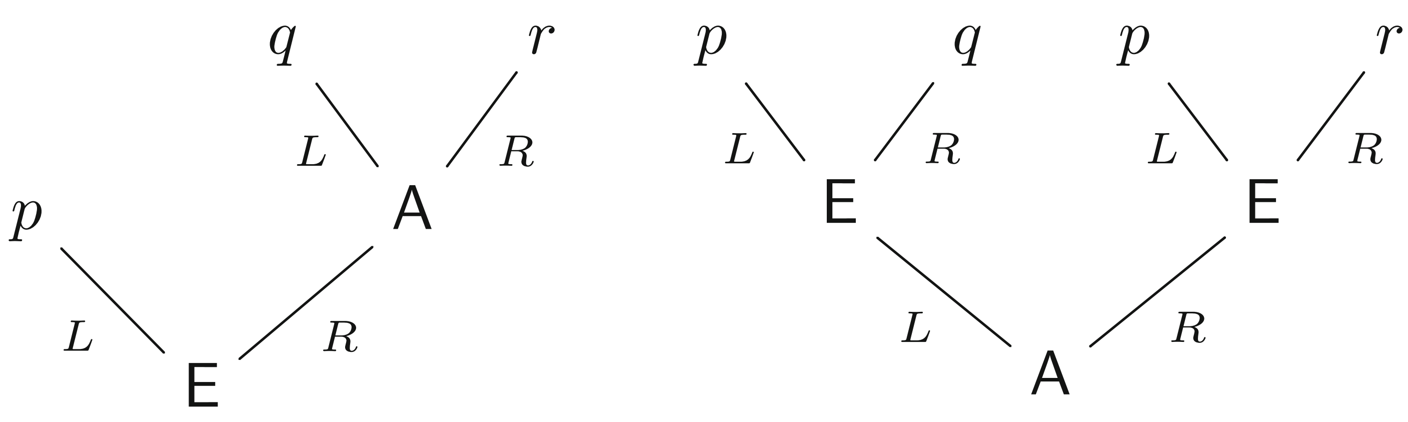
G. Bonanno (1992)

GAMES AS PROCESSES

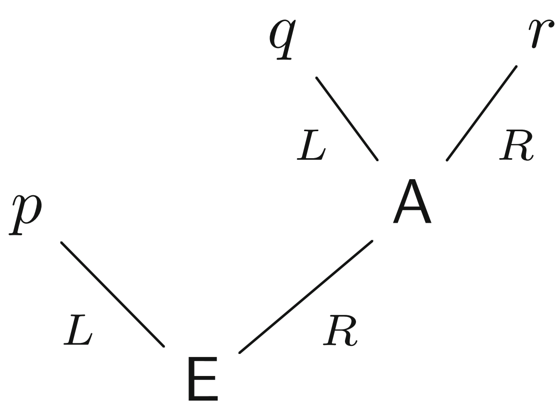
- Extensive-form games are natural process models.
- When are two processes the same?
- A modal logic perspective and the notion of bisimulation
- “When are two games the same?” ≈ “Do they exhibit the same properties that can be expressed in some appropriate language?”

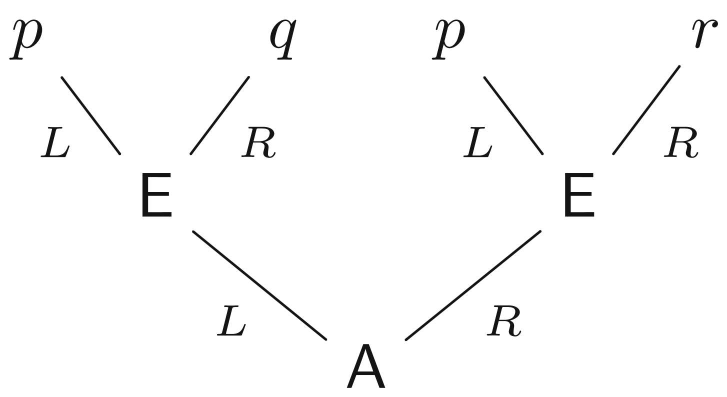
J. van Benthem (2002)

POWERS OF PLAYERS



POWERS OF PLAYERS





E: {p}, {p, q}, {p, r}, {q, r}
A: {p, q}, {p.r}

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GAMES BOARDS (GAME MODELS): POWERS OF PLAYERS

— — — — —o— — — — —

- Let B be a set of states or game positions (the game board).
- Each game g and player i is associated with a relation $F^i_g \subseteq B \times 2^B$: starting from a position in B player i can force the outcome of g to be among a subset of B.
    - Monotonicity: s F^i_g U and U $\subseteq$ V imply s F^i_g V
    - Consistency: s F^i_g U imply not s $F^{\bar{i}}_g$ (B $\setminus$ U)
- Two games g and h on the same game board are said to be equivalent (denoted by, $\mathrm{g} \approx$ h) if F^i_g = F^i_h for each i.

COMPOSITE GAME OPERATORS

- s $F^1_{g \cup g'}$ U iff s F^1_g U or s $F^1_{g'}$ U
- s $F^2_{g \cup g'}$ U iff s F^2_g U and s $F^2_{g'}$ U
- s $F^i_{g^d}$ U iff s $F^{\bar{i}}_g$ U
- s $F^i_{g;g'}$ U Iff there exists V: s F^i_g V and for all v in V, v $F^i_{g'}$ U

GAME LOGIC

— — — — —o— — — — —

- Models M : $(B, \{F_g^i : g \in \mathscr{G}\}, V)$, where, $V : \mathscr{P} \to 2^B$.
- Language L : $\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle G, i\rangle\varphi$; $G := g \mid G \cup G \mid G;G \mid G^d$.
- Here, $p \in \mathscr{P}$ and $g \in \mathscr{G}$, the set of atomic propositions and the set of atomic games, respectively.
- M, s ⊨ $\langle G, i\rangle\varphi$ iff there exists $U \subseteq B$ such that s F_g^i U and for all u in U, M, u ⊨ φ.
- Formulas: $\langle G \cup G',1\rangle\varphi \leftrightarrow \langle G,1\rangle\varphi \vee \langle G',1\rangle\varphi$, $\langle G^d, i\rangle\varphi \leftrightarrow \neg\langle G, \bar{i}\,\rangle\neg\varphi$

R. Parikh (1985)

GAME ALGEBRA

— — — — —o— — — — —

- Distributive Lattice (G, $\wedge$, $\vee$).
- De Morgan Laws: $-(x \wedge y) \approx (-x \vee -y)$, $-(x \vee y) \approx (-x \wedge -y)$
- Double negation: $- - x \approx x$
- $x; (y; z) \approx (x; y); z$, $(x \vee y); z \approx (x; z) \vee (y; z)$, $(x \wedge y); z \approx (x; z) \wedge (y; z)$, $-(x; y) \approx - x; - y$,
- If $x \preceq y$ then $x; z \preceq y; z$ (here, $x \preceq y$ is an abbreviation for the equation $x \vee y \approx y$)

J. van Benthem (2000)

COMPLETE AXIOM SYSTEM OF GAME ALGEBRA

V. Goranko (2003)

Y. Venema (2003)

PARALLEL GAME : POWERS OF PLAYERS

- Let B be a set of states or game positions (the game board).
- Each game g and player i is associated with a relation $F^i_g \subseteq B \times 2^{2^B}$: starting from a position in B player i can force the outcome of g to be among a set of subsets of B.
  - Monotonicity: s F^i_g U and U $\subseteq$ V imply s F^i_g V
  - Consistency: s F^i_g U imply not s $F^{\bar{i}}_g$ ($2^B \setminus$ U)
- Two games g and h on the same game board are said to be equivalent (denoted by, $\mathrm{g} \approx$ h) if F^i_g = F^i_h for each i.

J. van Benthem, S. Ghosh and F. Liu (2008)

PARALLEL GAME OPERATOR

- s $F^{1}_{g\cup g'}$ U iff s F^{1}_{g} U or s $F^{1}_{g'}$ U
- s $F^{2}_{g\cup g'}$ U iff s F^{2}_{g} U and s $F^{2}_{g'}$ U
- s $F^{i}_{g^d}$ U iff s $F^{\bar{i}}_{g}$ U
- s $F^{i}_{g;g'}$ U Iff there exists V: s F^{i}_{g} V and for all v in $\cup$ V, v $F^{i}_{g'}$ U
- s $F^{i}_{g\times g'}$ U Iff there exists Y, Z: s F^{i}_{g} Y and s $F^{i}_{g'}$ Z and X = {y $\cup$ z: y in Y and z in Z}

A TOY EXAMPLE

$$\begin{tikzcd} & E \arrow[dl] \arrow[dr] & \\ 1 & \mathbf{G} & 2 \end{tikzcd} \qquad\qquad \begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ 3 & \mathbf{H} & 4 \end{tikzcd}$$

A TOY EXAMPLE

$$\begin{tikzcd} & E \arrow[dl] \arrow[dr] & \\ 1 & \mathbf{G} & 2 \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ 3 & \mathbf{H} & 4 \end{tikzcd}$$

E: {{1}}, {{2}}

E: {{3}, {4}}

A TOY EXAMPLE

$$\begin{tikzcd} & E \arrow[dl] \arrow[dr] & \\ 1 & \mathbf{G} & 2 \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ 3 & \mathbf{H} & 4 \end{tikzcd}$$

E: {{1}}, {{2}}

E: {{3}, {4}}

E: {{1, 3}, {1, 4}}, {{2, 3}, {2, 4}}

PARALLEL GAME LOGIC

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- Models M : $(B, \{F_g^i : g \in \mathcal{G}\}, V)$, where, $V : \mathcal{P} \to 2^B$.
- Language L : $\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle G, i\rangle\varphi$; $G := g \mid G \cup G \mid G; G \mid G^d \mid G \times G$
- Here, $p \in \mathcal{P}$ and $g \in \mathcal{G}$, the set of atomic propositions and the set of atomic games, respectively.
- M, s ⊨ $\langle G, i\rangle\varphi$ iff there exists U ⊆ B such that s F_g^i U and for all u in ∪ U, M, u ⊨ φ.
- Formulas: $\langle G \cup G', 1\rangle\varphi \leftrightarrow \langle G, 1\rangle\varphi \vee \langle G', 1\rangle\varphi$, $\langle G^d, i\rangle\varphi \leftrightarrow \neg\langle G, \bar{i}\rangle\neg\varphi$, $\langle G \times G', i\rangle\varphi \leftrightarrow \langle G, i\rangle\varphi \wedge \langle G', i\rangle\varphi$

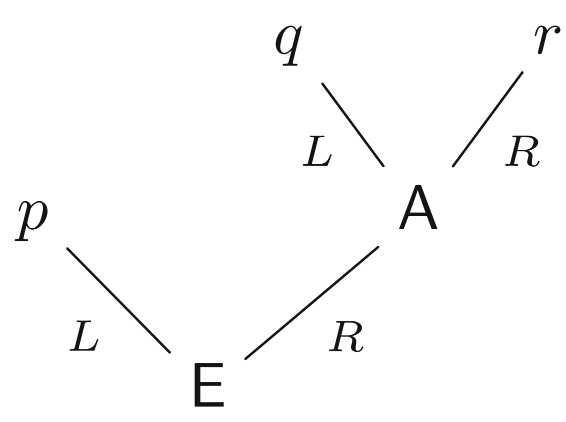
PARALLEL GAME ALGEBRA

— — — — —o— — — — —

- Game Algebra
- $x \times (y \times z) \approx (x \times y) \times z$ $\quad$ $x \times y \approx y \times x$
- $x \times (y \vee z) \approx (x \times y) \vee (x \times z)$ $\quad$ $x \times (y \wedge z) \approx (x \times y) \wedge (x \times z)$
- $-(x \times y) \approx -x \times -y$
- What else?

Q: What is a complete axiom system for this parallel game algebra?

POWERS OF PLAYERS: ONE MORE NOTION



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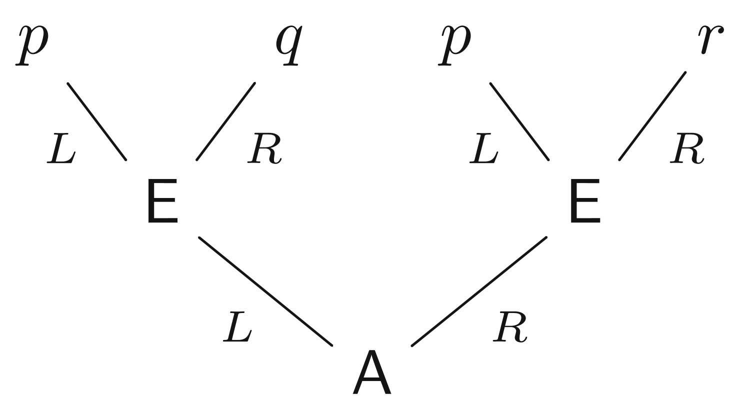

E: {p}, {p, q}, {p, r}, {q, r}
A: {p, q}, {p.r}

BASIC POWERS OF PLAYERS

— — — — —o— — — — —

- Let B be a set of states or game positions (the game board).
- Each game g and player i is associated with a relation $F_g^i \subseteq B \times 2^B$: starting from a position in B player i can force the outcome of g to be among a subset of B.
    - Consistency: s F_g^i U imply not s $F_g^{\bar{i}}$ (B $\setminus$ U)
    - Exhaustiveness: If s F_g^i U and u is in U, then there exists V: s $F_g^{\bar{i}}$ V and u is in V
- Two games g and h on the same game board are said to be equivalent (denoted by, $g \approx$ h) if F_g^i = F_h^i for each i.

J. Van Benthem, N. Bezhanishvili and S. Enqvist (2019)

COMPOSITE GAME OPERATORS

- s $F^1_{g\cup g'}$ U iff s F^1_g U or s $F^1_{g'}$ U
- s $F^2_{g\cup g'}$ U iff there exists X, Y: s F^2_g X and s $F^2_{g'}$ Y and U = X $\cup$ Y
- s $F^i_{g^d}$ U iff s $F^{\bar{i}}_g$ U

BASIC GAME ALGEBRA

— — — — —o— — — — —

- $x \cup y \approx y \cup x$ $\quad$ $x \cap y \approx y \cap x$
- $x \cup (y \cup z) \approx (x \cup y) \cup z$ $\quad$ $x \cap (y \cap z) \approx (x \cap y) \approx z$
- $- - x \approx x$
- $-(x \cap y) \approx (-x \cup -y)$ $\quad$ $-(x \cup y) \approx (-x \cap -y)$

Q: What is a complete axiom system for this basic game algebra?

TO END WITH ...

- There are many other similar open questions in terms of different game algebras
- One can consider different levels as well: adding preferences, knowledge, explicit strategies
- Other interesting notions of player powers

Thank you