Synthesis of Nash Equilibria and Subgame Perfect Equilibria in Games Played on Graphs

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ReLaX Workshop on Games

This talk

- General context
  - Systems composed of multiple interacting components (= players)
  - Each player has his own objective that is compatible or not with the objectives of the other players
  - Modelization with multiplayer non zero-sum games played on graphs
    - The interactions between the players are modelized by a graph
    - The objectives are qualitative or quantitative
  - Focus on the synthesis of equilibria in graph games, like Nash equilibria and subgame perfect equilibria

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    - The interactions between the players are modelized by a graph
    - The objectives are qualitative or quantitative
  - Focus on the synthesis of equilibria in graph games, like Nash equilibria and subgame perfect equilibria
- Study of two main problems
  - Does there always exist an equilibrium? Can we construct it?
  - Can we decide (algorithm) the existence of an equilibrium under some constraints? Complexity class of this decision problem?

This talk

Introductory survey with presentation of some classical and some recent results, including results from UMONS team

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More details

- in the book chapter "Solution Concepts and Algorithms for Infinite Multiplayer Games" [GU08]
- in my survey "Computer Aided Synthesis: a Game Theoretic Approach" in the Proceedings of DLT 2017 [Bru17]
- in the survey "Non-Zero Sum Games for Reactive Synthesis" in the Proceedings of LATA 2016 [BCH+16]

Multiplayer non zero-sum games

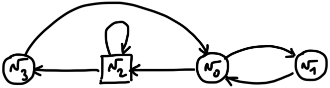
Definition

Arena $A = (\Pi, V, (V_i)_{i \in \Pi}, E)$ with

- finite set Π of players
- finite sets V of vertices and E of edges with no deadlock
- partition $(V_i)_{i \in \Pi}$ of V with V_i controlled by player $i \in \Pi$

Paths in G

- Play: infinite path
  $\rho = \rho_0 \rho_1 \ldots \in V^\omega$
- History: finite path
  $h = \rho_0 \rho_1 \ldots \rho_n \in V^*$



Player 1 ◯, Player 2 □

Multiplayer non zero-sum games

Definition

Game $G = (A, (f_i)_{i \in \Pi})$ with

- arena A
- for each player $i \in \Pi$, payoff function $f_i : Plays \to \mathbb{R}$

Player i prefers play ρ to play ρ' if $f_i(\rho) > f_i(\rho')$

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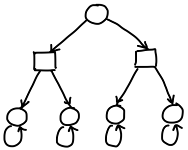
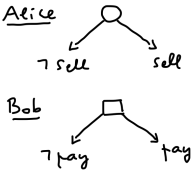
Example: Exchange protocol [CDFR17]





A prefers plays $2, 4$ (payoff 1) to plays $1, 3$ (payoff 0)
B prefers plays $3, 4$ (payoff 1) to plays $1, 2$ (payoff 0)

Classical payoff functions - Boolean games

- f_i : $Plays \to \{0, 1\}$
- Objective Ω_i of player i: $\Omega_i = \{\rho \in Plays \mid f_i(\rho) = 1\}$

Classical payoff functions - Boolean games

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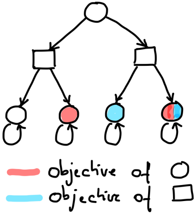
Definition

Some classical ω-regular objectives are:

- Reachability objective: visit a vertex of $U \subseteq V$ at least once
- Büchi objective: visit a vertex of U infinitely often
- Safety, Co-Büchi, Muller, Rabin, Streett, Parity

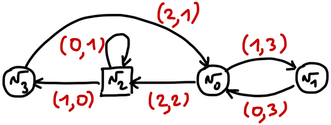
Example of exchange protocol

- Reachability objective for both players
- Play 3 with payoff $(0, 1)$



Classical payoff functions - Quantitative games

- Arena extended with weight functions $(w_i)_{i \in \Pi}$ such that $w_i : E \to \mathbb{Q}$
- f_i : $Plays \to \mathbb{R}$ defined from w_i
  Player i prefers to maximize payoff $f_i(\rho)$



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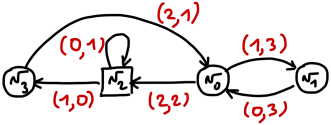


Definition

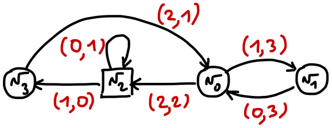
Classical payoff $f_i(\rho)$ of a play $\rho = \rho_0\rho_1\rho_2\ldots$

- $\text{Sup}_i(\rho) = \sup_{n \in \mathbb{N}} w_i(\rho_n, \rho_{n+1})$
- $\text{LimSup}_i(\rho) = \limsup_{n \to \infty} w_i(\rho_n, \rho_{n+1})$
- Mean-payoff $\overline{\text{MP}}_i(\rho) = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} w_i(\rho_k, \rho_{k+1})$
- Discounted-sum $\text{Disc}_i^{\lambda}(\rho) = \sum_{k=0}^{\infty} w_i(\rho_k, \rho_{k+1})\lambda^k$, where $\lambda \in \,]0, 1[$
- Variants: $\text{Inf}_i(\rho), \text{LimInf}_i(\rho), \underline{\text{MP}}_i(\rho)$

Classical payoff functions - Quantitative games

Example

- LimSup_1 for player 1
- $\overline{\text{MP}}_2$ for player 2
- Play $\rho = (v_0 v_1)^\omega$ with payoff $(1, 3)$



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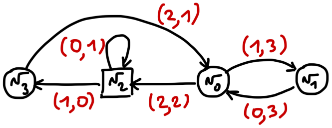
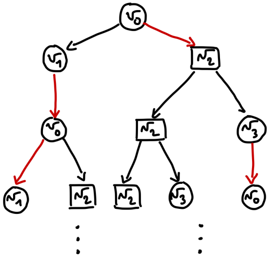


In the sequel

- Boolean games: games with ω-regular objectives
- Quantitative games: games with payoff functions Sup, Inf, LimSup, LimInf, Mean-payoff, Discounted-sum

Strategies

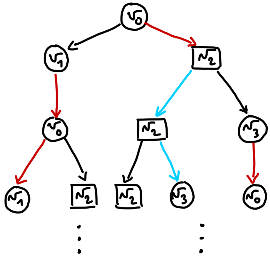
Unravelling of G from v_0



Strategy for player i:
function $\sigma_i : V^* V_i \to V$ such that
$\sigma_i(hv) = v'$ with $(v, v') \in E$

Strategy profile $(\sigma_i)_{i \in \Pi}$

- with outcome $\rho = \langle(\sigma_i)_{i\in\Pi}\rangle_{v_0}$ from initial vertex v_0
- with payoff $(f_i(\rho))_{i\in\Pi}$



1 Multiplayer non zero-sum games

2 Nash equilibria

3 Subgame perfect equilibria

4 Conclusion

Nash equilibria

Classical notion such that

- each player wants to maximize his payoff
- he is indifferent to the payoff of the other players

Nash equilibria

Classical notion such that

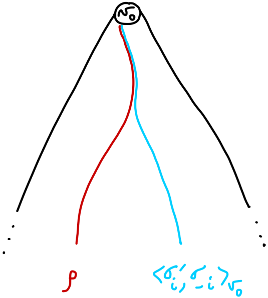
- each player wants to maximize his payoff
- he is indifferent to the payoff of the other players

Definition [Nas50]

The strategy profile $(\sigma_i)_{i\in\Pi}$ with outcome ρ from v_0 is a Nash equilibrium (NE) if, for each player $i \in \Pi$, for each strategy σ_i' of i,

$$f_i(\rho) \quad \geq \quad f_i(\langle \sigma_i', \sigma_{-i} \rangle_{v_0})$$

Notation: $\sigma_{-i} = (\sigma_j)_{j\in\Pi\setminus\{i\}}$



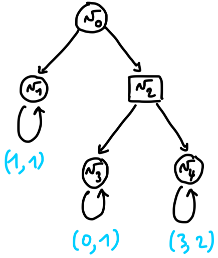
Informally, $(\sigma_i)_{i\in\Pi}$ is an NE if no player has an incentive to deviate from his strategy, if the other players stick to their own strategies

Nash equilibria

Example

Simple game

- with 3 plays
- and their payoffs indicated below



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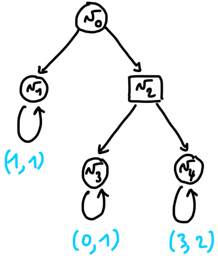
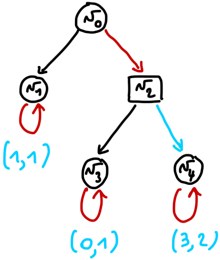


- NE with outcome $v_0 v_2 v_4^\omega$ with payoff $(3, 2)$
- No incentive to deviate:
  - If player 1 deviates to v_1, he will get 1 instead of 3
  - If player 2 deviates to v_3, he will get 1 instead of 2



Existence results for NEs

Theorem

- In Boolean games, there always exists an NE (even in games with Borel objectives) [CMJ04]
- In quantitative games, there always exists an NE [BDS13]

Existence results for NEs

General proof technique[1]:

Given a multiplayer non zero-sum game G

- given player i, coalition $-i$ of all the other players that is opposed to i
- two-player zero-sum game G_i where player i wants to maximize $f_i(\rho)$ while player $-i$ wants to minimize it

[1] Well-known method in game theory (in Folk Theorem in repeated games) [OR94]

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Definition In G_i

- A vertex v has a value $val_i(v)$ if
  - player i has a strategy from v to ensure a payoff $\geq \mathrm{val}_i(v)$
  - player $-i$ has a strategy from v to ensure a payoff $\leq \mathrm{val}_i(v)$
- Those strategies are called optimal

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Well-known results (see [Bru17]) : Boolean and quantitative two-player zero-sum games G_i all have optimal strategies (algorithms - complexity)

[1] Well-known method in game theory (in Folk Theorem in repeated games) [OR94]

Existence results for NEs

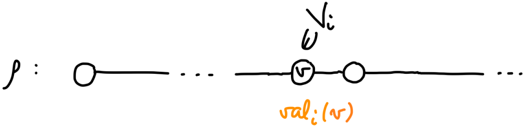
Nice characterization of NE outcomes

Theorem [GU08, UW11], [Bru17]

Let G be a game such that for all i

- the payoff function f_i is prefix-independent (i.e. $f_i(\rho) = f_i(h\rho)$ $\forall h, \rho$)
- the game G_i has optimal strategies τ_i and τ_{-i} for both players

Then ρ is the outcome of an NE iff $f_i(\rho) \geq \text{val}_i(v)$ whenever v is a vertex of ρ with $v \in V_i$



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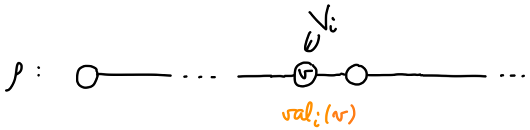

Remark: Many Boolean and quantitative games have prefix-independent payoff functions

Existence results for NEs

Corollary [BDS13]

Let G be a multiplayer non zero-sum game such that for all i

- the payoff function f_i is prefix-independent
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Then one can construct an NE in G

Existence results for NEs

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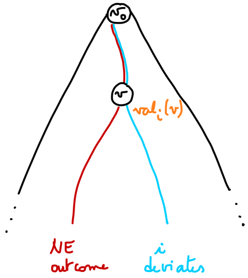
Construction of the NE:

- play as τ_i for each player i
  (player i plays selfishly and optimally with respect to f_i)
- as soon as some player i deviates, punish i by playing τ_{-i}
  (coalition $-i$ plays against player i with respect to f_i)



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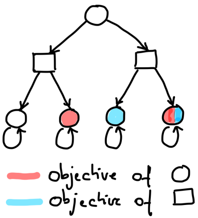
- Corollary: Boolean and quantitative games with prefix-independent payoff functions admit an NE
- NE characterization under more general hypotheses in [BDS13]: NE existence (effective construction) for all Boolean and quantitative games

Constrained existence for NEs

NEs with low payoffs may coexist with more interesting NEs



Example of exchange protocol

- NE with payoff $(0,0)$ (play 1)
- NE with payoff $(1,1)$ (play 4)

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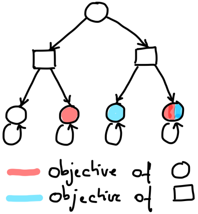

Decision problem

- Given thresholds $(\mu_i)_{i \in \Pi}, (\nu_i)_{i \in \Pi} \in (\mathbb{Q} \cup \{-\infty, +\infty\})^{|\Pi|}$, decide whether there exists an NE with outcome ρ such that $\mu_i \leq f_i(\rho) \leq \nu_i$ for all i

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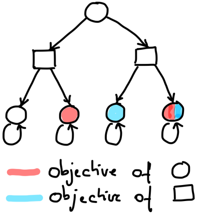

Decision problem

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- For Boolean games, $(\mu_i)_{i\in\Pi}, (\nu_i)_{i\in\Pi} \in \{0,1\}^{|\Pi|}$
  - $\mu_i = 1 \rightarrow$ player i must satisfy his objective Ω_i
  - $\nu_i = 0 \rightarrow$ player i cannot satisfy his objective Ω_i

Constrained existence for NEs

General proof technique

- Based on the previous characterization of NE outcomes
- For prefix-independent f_i, study of the existence of plays ρ such that
  1. $f_i(\rho) \geq \mathrm{val}_i(v)$ for all v, i such that v is a vertex of ρ with $v \in V_i$
  2. $\mu_i \leq f_i(\rho) \leq \nu_i$ for all i

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Theorem

- In Boolean games [Umm08, CFGR16], the constrained NE existence problem is
  - Büchi, Muller: P-complete
  - Reachability, Safety, co-Büchi, Parity, Streett: NP-complete
  - Rabin: in P^{NP}, NP- and co-NP-hard
- In quantitative games [Umm08, UW11, CFGR16], the constrained NE existence problem is
  - LimSup: P-complete
  - Sup, Inf, LimInf, Mean-payoff: NP-complete
  - Discounted-sum: open

Subgame perfect equilibria

NEs have some drawbacks

- They do not take into account the sequential nature of games played on graphs
- They are subject to uncredible threat

Subgame perfect equilibria

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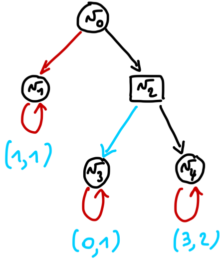
Example

- Player 1 will not deviate, due to the threat of player 2
- Uncredible threat of player 2
- More rational for him to go to v_4 in the subgame induced by v_2, v_3, v_4



Subgame perfect equilibria

Classical notion: strategy profile that is an NE after every history of the game, and not only from v_0

Subgame perfect equilibria

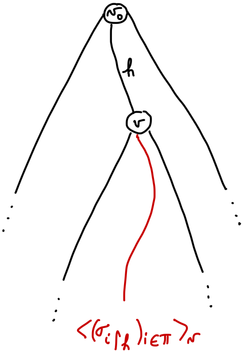
Classical notion: strategy profile that is an NE after every history of the game, and not only from v_0

Definition [Sel65]

The strategy profile $(\sigma_i)_{i \in \Pi}$ is a subgame perfect equilibrium (SPE) from v_0 if $(\sigma_{i \upharpoonright h})_{i \in \Pi}$ is an NE in $G_{\upharpoonright h}$ from v, for every history hv of G

Notation

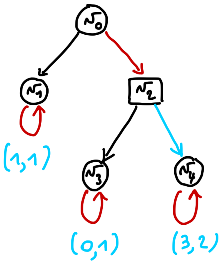
- Subgame $G_{\upharpoonright h}$ with initial vertex v after history h
- Strategy $\sigma_{i \upharpoonright h}$ in $G_{\upharpoonright h}$ induced by σ_i after history h



Subgame perfect equilibria

Example of an SPE

- NE outcome $v_0 v_2 v_4^\omega$ in the game G from v_0
- NE outcome $v_2 v_4^\omega$ in the subgame $G_{\upharpoonright v_0}$ from v_2
- NE outcome at each subgame which is a “leaf”



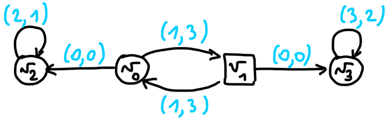
Existence results for SPEs

Theorem

- In Boolean games, there always exists an SPE (even in games with Borel objectives) [GU08]
- There exist quantitative games with no SPE [SV03]

Example with no SPE

Mean-payoff or LimSup
(= payoff of the ending cycle)



Existence results for SPEs

Theorem

In games with payoff functions $(f_i)_{i \in \Pi}$, there always exists an SPE

1. in games played on a finite tree [Kuh53]
2. if each f_i is bounded and continuous [FL83, Har85]
3. if each f_i has finite range and is upper-semicontinuous[2] [FKM+10]

[2] whenever $\lim_n \rho_n = \rho$, then $\limsup_n f_i(\rho_n) \leq f_i(\rho)$

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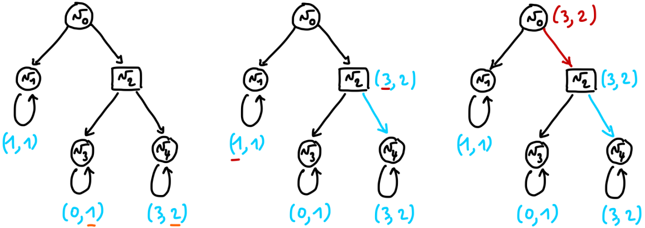
Proof of 1: Backward induction from the leaves to the root



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- Corollary of 2: Existence of an SPE for quantitative games with Discounted-sum functions f_i
- Proof techniques: Not known characterization of SPE outcomes, as for NE outcomes (except for some particular classes of games, see my second talk)

[2] whenever $\lim_n \rho_n = \rho$, then $\limsup_n f_i(\rho_n) \leq f_i(\rho)$

Constrained existence for SPEs

Few results

Theorem

In Boolean games, the constrained SPE existence problem is

- Reach: PSPACE-complete [BBGR18]
- Parity: in EXPTIME, NP-hard [Umm06, GU08]

In quantitative games, the constrained SPE existence problem is

- Quantitative reach: PSPACE-complete [BBG+19]

Constrained existence for SPEs

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In quantitative games, the constrained SPE existence problem is

- Quantitative reach: PSPACE-complete [BBG+19]

Proof technique for reachability

- notion of weak SPE, see next slides
- see also my next talk

Weak SPEs

Definition [BBMR15]

- Weak SPE: variant of SPE such that when one player deviates, he can only use one-shot deviating strategies
- A strategy σ'_i is one-shot deviating from a strategy σ_i if σ'_i and σ_i only differ at the initial vertex

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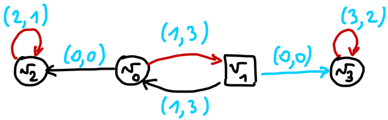
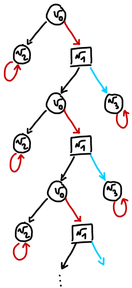
Example (continued)



Weak SPE with outcome $v_0 v_1 v_3^\omega$ and payoff $(3, 2)$



Weak SPEs

Proposition

- SPEs and weak SPEs are equivalent notions
  - in games played on a finite tree [Kuh53]
  - in games with payoff functions f_i that are continuous or even upper-semicontinuous [BBMR15]
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Comments

- In [Kuh53], one-shot deviation property (equivalent to the notion of weak SPE)
- Weaks SPEs are equivalent to SPEs for several large classes of games
- They are much easier to manipulate
- They have been further studied in [BRPR17, BBGR18, Goe20]

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Thank you!

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