Nash Equilibria and Subgame Perfect Equilibria in Reachability Games

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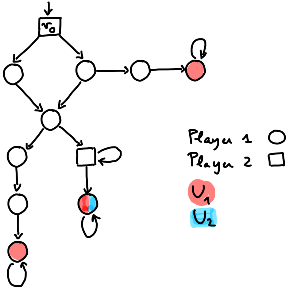
ReLaX Workshop on Games

Reachability games

- Multiplayer non zero-sum games played on graphs
- Boolean reachability games: each player i has a target set U_i of vertices that he wants to reach
- Quantitative reachability games: each player i has a target set U_i of vertices that he wants to reach as quicky as possible (number of edges to reach U_i)



Reachability games

From the previous talk

Theorem

In Boolean reachability games,

- there always exists an NE [CMJ04] (resp. SPE [GU08])
- the constrained existence problem for NEs is NP-complete [CFGR16] (resp. for SPEs is PSPACE-complete [BBGR18])

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Here

Theorem

In quantitative reachability games,

- there always exists an NE (resp. SPE) [FL83, Har85]
- the constrained existence problem for NEs is NP-complete [BBGT19] (resp. for SPEs is PSPACE-complete [BBG+19])

Reachability games

Approach

- Characterization
  - Well-known approach for NEs: characterization of NE outcomes
  - New approach for SPEs: characterization of SPE outcomes

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- Weak SPEs
  - SPEs and weak SPEs are equivalent notions for quantitative games

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Let's play!

- first in two-player zero-sum games
- then in multi-player non zero-sum games

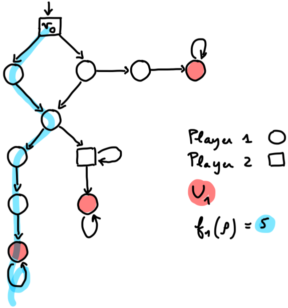
Reachability two-player zero-sum games

Definition

- Game $G = (A, U_1, f_1)$ played on an arena $A = (V, V_1, V_2, E)$
- Two players, $i = 1, 2$, such that Player i controls vertices in V_i
- Target set U_1 for Player 1 that he wants to reach as quickly as possible
- Given a play $\rho = \rho_0\rho_1\ldots$

$$\text{cost } f_1(\rho) = \begin{cases} \min\{k \in \mathbb{N} \mid \rho_k \in U_1\} \\ \quad \text{if } \rho \text{ visits } U_1 \\ \infty \text{ otherwise} \end{cases}$$

- Zero-sum: Player 2 has the opposite objective



Reachability two-player zero-sum games

Zero-sum games: $f_1(\rho)$ is a cost that

- Player 1 wants to minimize
- Player 2 wants to maximize

Reachability two-player zero-sum games

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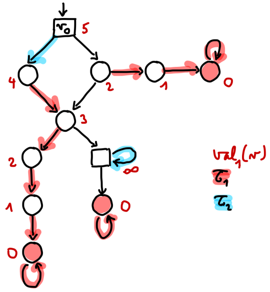
Theorem [GZ05]

In reachability two-player zero-sum games,

- Each vertex v has a value $\mathrm{val}_1(v)$
- There exist optimal strategies τ_i for each player i such that
  - With τ_1, player 1 can ensure a cost $\leq \mathrm{val}_1(v)$ from v
  - With τ_2, player 2 can ensure a cost $\geq \mathrm{val}_1(v)$ from v

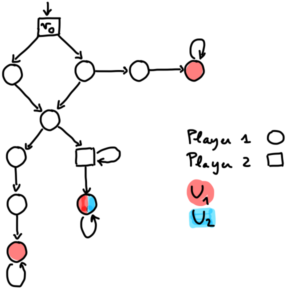
(with polynomial algorithms)



Reachability multi-player non zero-sum games

Definition

- Game $G = (A, (U_i)_{i \in \Pi}, (f_i)_{i \in \Pi})$ played on an arena $A = (\Pi, V, (V_i)_{i \in \Pi}, E)$
- For each player i, target set U_i and related function f_i
- Non zero-sum game



The players are no longer adversarial. They want to reach their target set as quickly as possible

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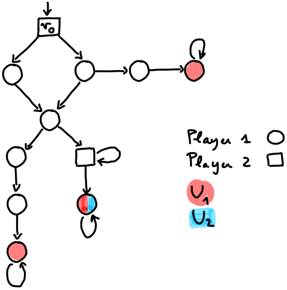

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Well-studied solution concepts:

- Nash equilibrium
- Subgame perfect equilibrium

Nash equilibria

Definition [Nas50]

The strategy profile $(\sigma_i)_{i \in \Pi}$ with outcome ρ from v_0 is a Nash equilibrium (NE) if no player has an incentive to unilaterally deviate (to decrease $f_i(\rho)$)

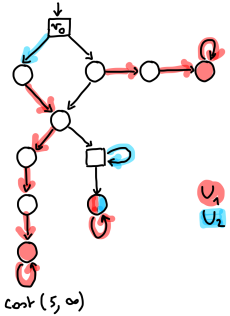
Nash equilibria

Definition [Nas50]

The strategy profile $(\sigma_i)_{i \in \Pi}$ with outcome ρ from v_0 is a Nash equilibrium (NE) if no player has an incentive to unilaterally deviate (to decrease $f_i(\rho)$)

Example

- NE (σ_1, σ_2) from v_0 with cost profile $(5, \infty)$
- Player 1 has no incentive to deviate as he will get cost ∞ instead of 5
- Player 2 has no incentive to deviate as he will get the same cost ∞



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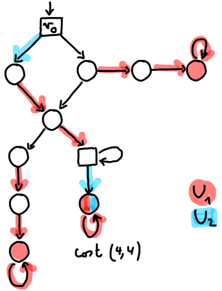
The strategy profile $(\sigma_i)_{i \in \Pi}$ with outcome ρ from v_0 is a Nash equilibrium (NE) if no player has an incentive to unilaterally deviate (to decrease $f_i(\rho)$)

- NE (σ_1, σ_2) from v_0 with cost profile $(5, \infty)$
- another NE (σ'_1, σ'_2) from v_0 with better cost profile $(4, 4)$



Nash equilibria

Problems

1. Existence: Does there always exist an NE?
2. Constrained existence: Given $(\mu_i)_{i \in \Pi} \in (\mathbb{N} \cup \{\infty\})^{|\Pi|}$, can we decide whether there exists an NE with outcome ρ such that $f_i(\rho) \leq \mu_i$ for all $i \in \Pi$?

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Answers

1. Yes [FL83, Har85]
2. Yes, NP-complete problem [BBGT19]

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Problem 1

- Each f_i function can be supposed to be continuous

$$f_i'(\rho) = 1 - \frac{1}{f_i(\rho) + 1} \text{ if } f_i(\rho) < +\infty, \text{ and } f_i'(\rho) = 1 \text{ otherwise.}$$

- Theorem [FL83, Har85]: If each f_i is bounded and continuous, then there exists an NE

Characterization of NE outcomes

Problem 2 is based on the following characterization

Definition

- For each $v \in V_i$, let $\text{val}_i(v)$ be the lowest cost w.r.t f_i that player i can ensure against the coalition of the other players (see slide 8)

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- For each $v \in V_i$, let $\mathsf{val}_i(v)$ be the lowest cost w.r.t f_i that player i can ensure against the coalition of the other players (see slide 8)
- A play $\rho = \rho_0\rho_1\ldots$ is val-consistent if for all k, if $\rho_k \in V_i$, then the suffix $\rho_{\geq k} = \rho_k\rho_{k+1}\ldots$ respects the constraint imposed by $\mathsf{val}_i(\rho_k)$:

$$f_i(\rho_{\geq k}) \leq \mathsf{val}_i(\rho_k)$$

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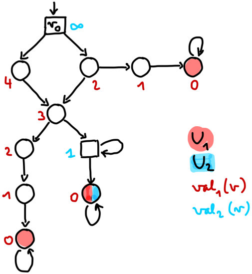
Theorem [BBGT19]

A play ρ is the outcome of an NE iff ρ is val-consistent

Characterization of NE outcomes

Example



Characterization of NE outcomes

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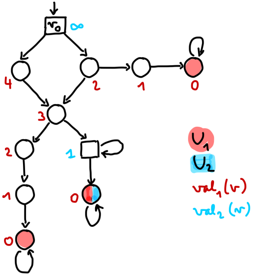

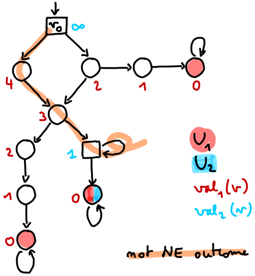

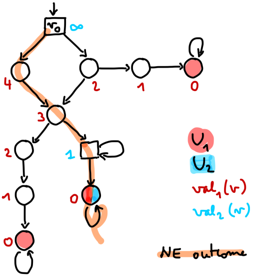

Characterization of NE outcomes

Proposition

- If there is an NE whose outcome satisfies the constraints, then there is one whose outcome is a lasso hg^ω with length $|hg| \leq (|\Pi| + 1) \cdot |V|$

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Solution to Problem 1 [BDS13]

- Construction of a simple NE thanks to the characterization

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Solution to Problem 1 [BDS13]

- Construction of a simple NE thanks to the characterization

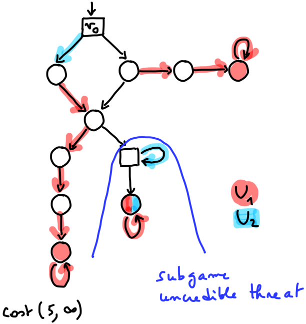
Solution to Problem 2 [BBGT19]

- Algorithm in NP
  - Guess a lasso as above
  - Check whether it satisfies the constraints
  - Check whether it is an NE outcome (thanks to the NE characterization - The values $\mathrm{val}_i(v)$ are computable in polynomial time)

Subgame perfect equilibria

Uncredible threat

- Previous NE example
- Player 1 has no incentive to deviate due to an uncredible threat of Player 2
- Refined solution concept: subgame perfect equilibrium



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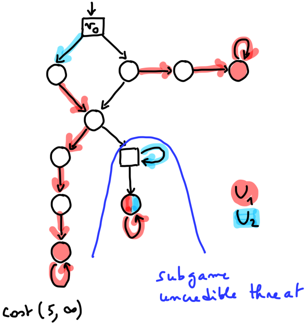

Definition [Sel65]

The strategy profile $(\sigma_i)_{i \in \Pi}$ is a subgame perfect equilibrium (SPE) from v_0 if it is an NE in each subgame $G_{\restriction h}$ from v, for every history hv of G

Subgame perfect equilibria

Theorem

In reachability multi-player non zero-sum games,

1. Existence: There always exists an SPE [FL83, Har85]
2. Constrained existence: Given $(\mu_i)_{i \in \Pi}$, deciding whether there exists an SPE with outcome ρ such that $f_i(\rho) \leq \mu_i$ for all $i \in \Pi$ is a PSPACE-complete problem [BBG+19]

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- Result 1: same argument as for NE
- Result 2: consequence of several papers
  - Inspiring paper [FKM+10]
    - Characterization of SPE outcomes for a particular class of games
    - Importance of weak SPEs
  - The constrained existence problem is decidable [BBMR15]
  - The constrained existence problem is PSPACE-complete [BBG+19]

Subgame perfect equilibria

- Weak SPE: variant of SPE where in each subgame, only one-shot deviating strategies at the initial vertex are allowed
- Weak SPEs and SPEs are equivalent notions for reachability games

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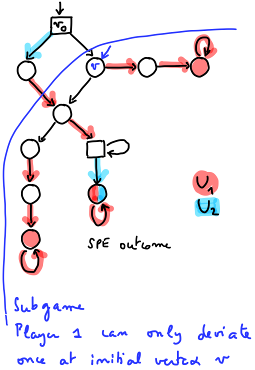
Example



Characterization of SPE outcomes

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Main idea

- compute λ^* iteratively: $\lambda_0, \lambda_1, \ldots, \lambda_k, \ldots \to \lambda^*$
- notion of λ_k-consistent play at step k

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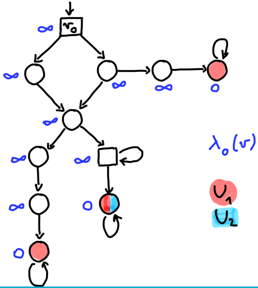
- compute λ^* iteratively: $\lambda_0, \lambda_1, \ldots, \lambda_k, \ldots \to \lambda^*$
- notion of λ_k-consistent play at step k

- Initially

  Let $v \in V_i$, then

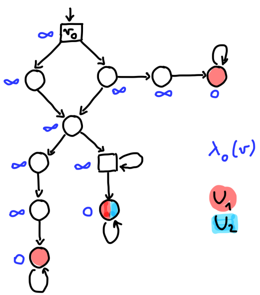
$$\lambda_0(v) = \begin{cases} 0 & \text{if } v \in U_i \\ \infty & \text{otherwise} \end{cases}$$

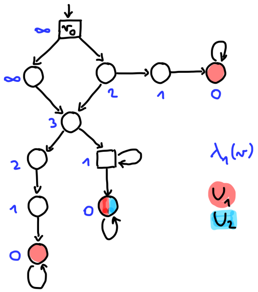
  All plays are λ_0-consistent



Characterization of SPE outcomes

- Update





Characterization of SPE outcomes

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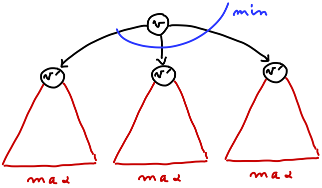
  Let $v \in V_i$, then

$$\lambda_{k+1}(v) = \begin{cases} 0 & \text{if } v \in U_i \\ 1 + \min_{v' \in succ(v)} \max\{\ f_i(v'\rho) \mid v'\rho \text{ is } \lambda_k\text{-consistent }\} \end{cases}$$



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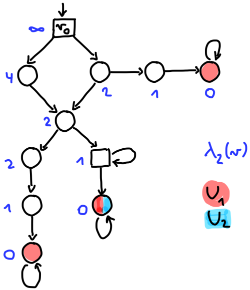

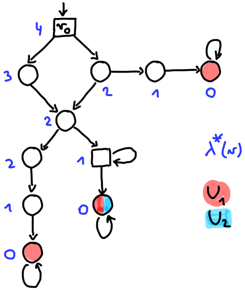

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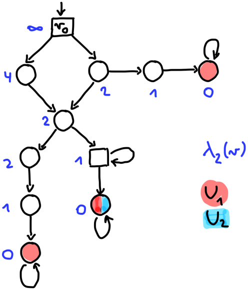

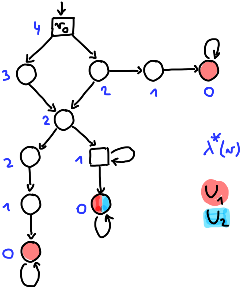

Theorem [BBG+19]

A play ρ is the outcome of an SPE iff ρ is λ^*-consistent

Constrained existence problem

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Carefully designed algorithm:

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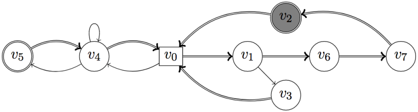
Carefully designed algorithm:

- More complex situation than in the previous example



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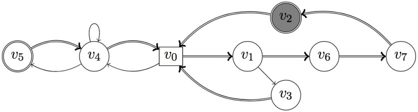

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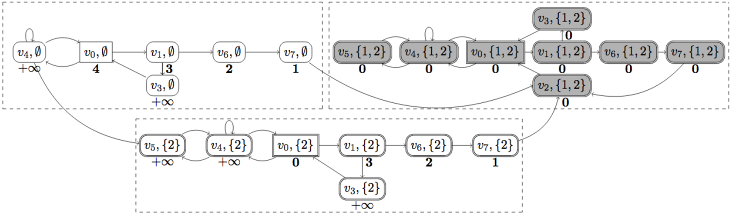
- Need to work with an extended game of exponential size



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- Theorem: Values for λ^* exponential in the size of G

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Theorem [BBG+19]

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Carefully designed algorithm:

- Theorem: Values for λ^* exponential in the size of G
- NPSPACE = PSPACE
  Guess a play (lasso) and test that it is λ^*-consistent and satisfies the constraints while computing the labeling function λ^* on the fly

Conclusion

Study of multi-player games with quantitative reachability objectives

- Existence
- Constrained existence

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Presentation of two notions of equilibria

- Nash equilibria: nice characterization of NE outcomes
- Subgame perfect equilibria and its weak version:
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Related work

- Other kinds of relevant NEs and SPEs in games with reachability objectives studied in [BBGT19]
- Very recent characteriation of SPE outcomes under general hypotheses (like for NE) by Jean-François Raskin's team

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