Introduction to Reinforcement Learning

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Supervised Learning - Classification / Prediction from training data

Unsupervised Learning - "Extract" information

Clustering

Reinforcement Learning - Sequence of steps

Playing a game

Repeatedly play the game

Learn a good strategy from outcome

Andrew Barto & Richard Sutton

Turing Award 2024

Book, 2nd ed 2018

An alternative approach to learning

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- Examples
  - Playing games — AlphaGo, reward is result of the game
  - Motion planning — robot searching for an optimal path with obstacles
  - Feedback control — balancing an object

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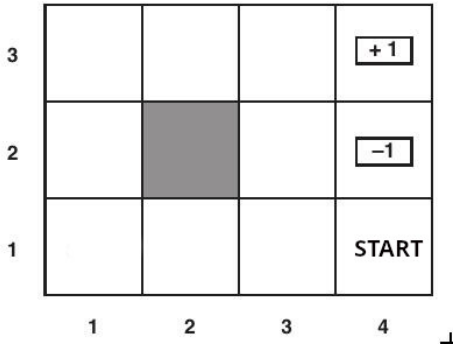
- Environment Model How the environment will behave
  - Given a state and action, what is the next state, reward?
  - Probabilistic, in general
  - Use models for *planning*
  - Can also use RL without models, trial-and-error learners

Motion planning example

- 4×3 grid
- Rewards are attached to states
  - Two terminal states with rewards $+1$, -1
  - All other states have reward -0.04
  - Move till you reach a terminal state
  - Maximize the sum of the rewards seen



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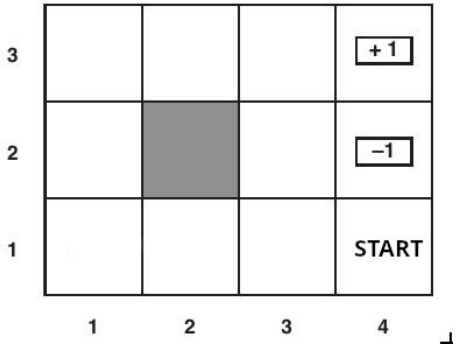

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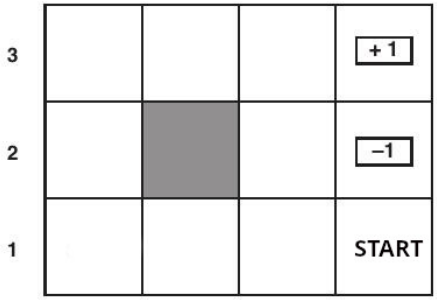
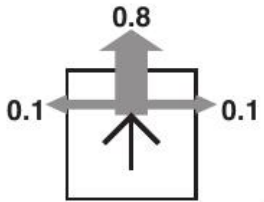
- Policy — which direction to move from a given square in the grid
- Outcome of action is nondeterministic
  - With probability 0.8, go in intended direction
  - With probability 0.2, deflect at right angles
  - Collision with boundary keeps you stationary





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- Optimal policy learned by repeatedly moving on the board
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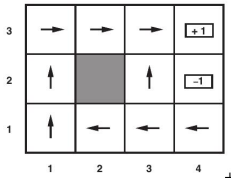

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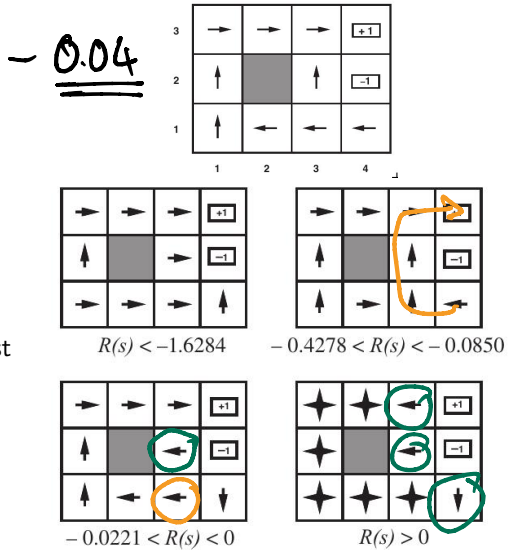
- Optimal policies for different value of $R(s)$, reward for non-final states
  - If $R(s) < -1.6284$, terminate as fast as possible
  - If $-0.4278 < R(s) < -0.0850$, risk going past -1 to reach $+1$ quickly
  - If $-0.0221 < R(s) < 0$, take no risks, avoid -1 at all cost
  - If $R(s) > 0$ avoid terminating



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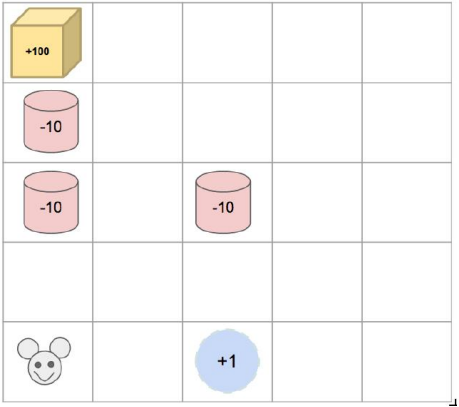

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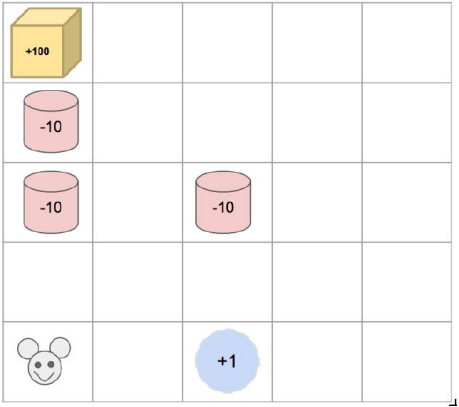

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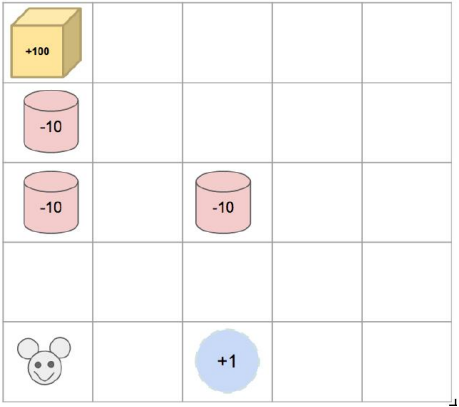
- Formalize these ideas using Markov Decision Processes



Bandits

- One-armed bandit — slang for a slot machine in a casino
  - Put in a coin and pull a lever (the arm)
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  - If we knew $q_*(a)$ we would always choose $A_t = \arg\max_a q_*(a)$
  - Assume $q_*(a)$ is unknown — build an estimate $Q_t(a)$ of $q_*(a)$ at time t

Exploration and exploitation

- Build $Q_t(a)$, estimate of $q_*(a)$ at time t, from past observations (sample average)

$$\frac{\text{sum of rewards when } a \text{ taken prior to } t}{\text{number of times } a \text{ taken prior to } t} = \frac{\sum_{i=1}^{t-1} R_i \cdot \mathbb{1}_{A_t=a}}{\sum_{i=1}^{t-1} \mathbb{1}_{A_t=a}}$$

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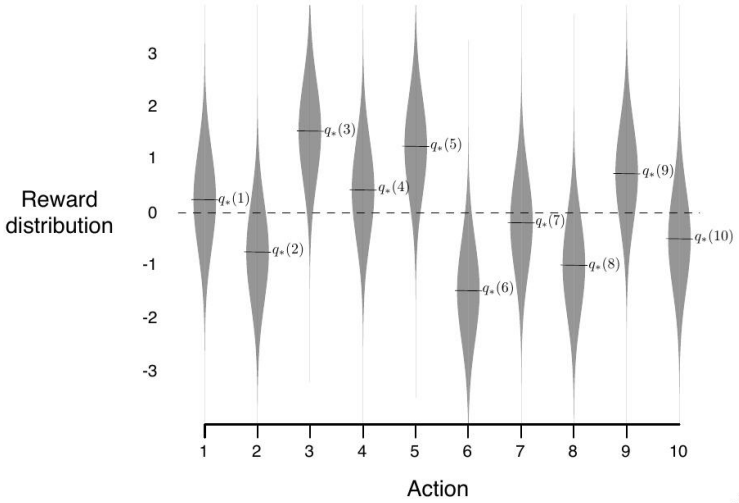
- ε-greedy is a simple way to balance exploitation with exploration
  - Theoretically, explores all actions infinitely often
  - Practical effectiveness depends

Exploration and exploitation

10 bandit experiment

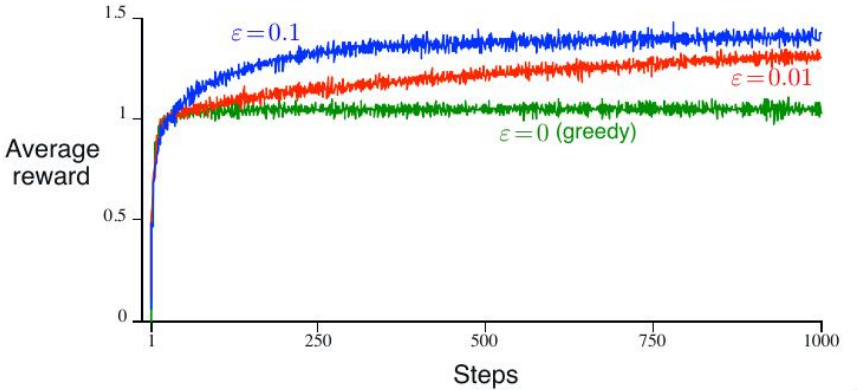
- Each bandit's reward follows Gaussian distribution
- Same variance, mean is chosen randomly



Exploration and exploitation

Performance of ε-greedy strategies

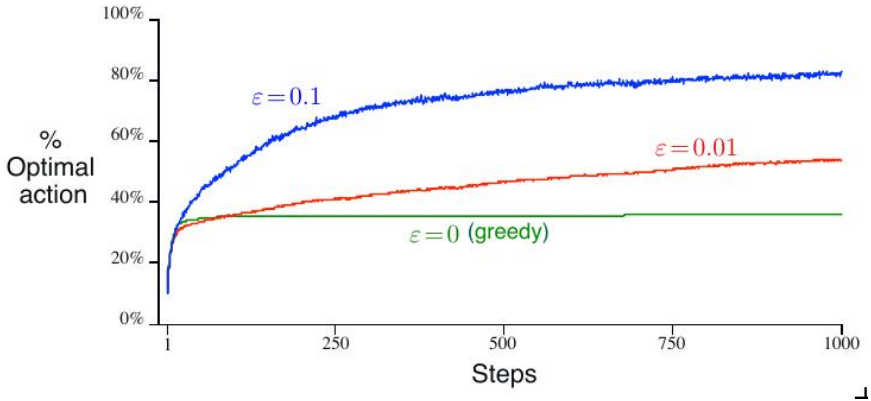
- Pure greedy strategy is sub-optimal
- Initial "learning rate" is more or less equal



Exploration and exploitation

Discovery of optimal actions

- Pure greedy strategy discovers optimal action only 1/3 of the time



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- We will see this pattern often:

  NewEstimate = OldEstimate + Step [Target - OldEstimate]

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$$= (1-\alpha)^n Q_1 + \sum_{i=1}^{n} \alpha(1-\alpha)^{n-i} R_i$$

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- $$\begin{aligned} Q_{n+1}^{a} &= Q_n + \alpha[R_n - Q_n] \quad = \alpha R_n + (1-\alpha)Q_n \\ &= \alpha R_n + (1-\alpha)[\alpha R_{n-1} + (1-\alpha)Q_{n-1}] \\ &= \alpha R_n + \alpha(1-\alpha)R_{n-1} + (1-\alpha)^2 Q_{n-1} \\ &= \alpha R_n + \alpha(1-\alpha)R_{n-1} + \alpha(1-\alpha)^2 R_{n-2} + \cdots + \alpha(1-\alpha)^{n-1} R_1 + (1-\alpha)^n Q_1 \end{aligned}$$

  $$= (1-\alpha)^n Q_1 + \sum_{i=1}^{n} \alpha(1-\alpha)^{n-i} R_i$$

- Exponentially decaying weighted average of rewards

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- Exponentially decaying weighted average of rewards
- Initial value Q_1 affects the calculation — different heuristics possible

Summary

- k-armed bandit is the simplest interesting situation to analyze
- ε-greedy strategy balances exploration and exploitation
- Incremental update rule for estimates

  NewEstimate = OldEstimate + Step [Target - OldEstimate]
- Exponentially decaying weighted average when rewards change over time (non-stationary)

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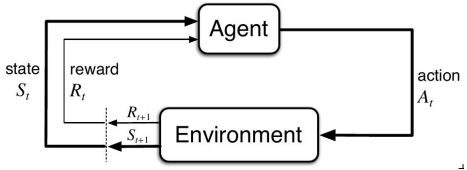

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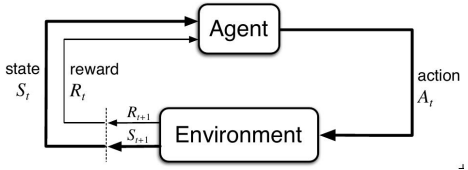

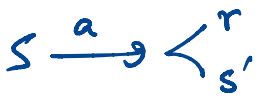

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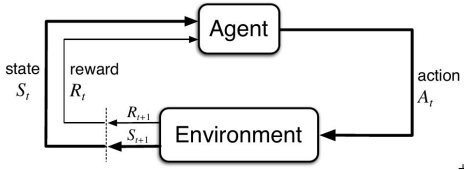

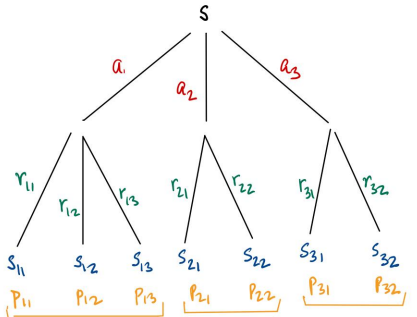

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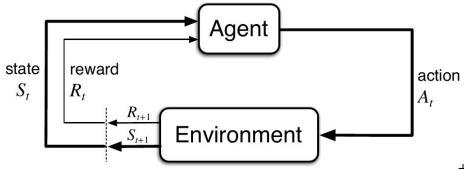
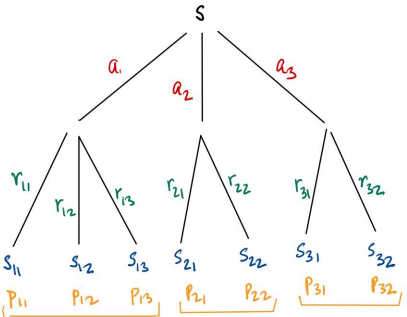
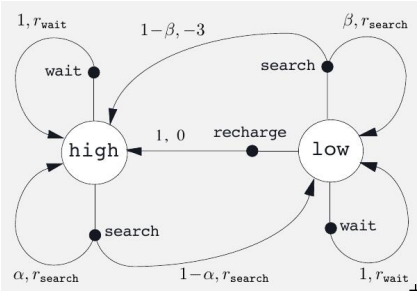
  - Backup diagram
- Typically assume finite MDPs — S, A and R are finite





MDP Example: Robot that collects empty cans

- State — battery charge: high, low
- Actions: search for a can, wait for someone to bring can, recharge battery
  - No recharge when high



s	a	s'	$p(s' \mid s, a)$	$r(s, a, s')$
high	search	high	α	r_{search}
high	search	low	$1-\alpha$	r_{search}
low	search	high	$1-\beta$	-3
low	search	low	β	r_{search}
high	wait	high	1	r_{wait}
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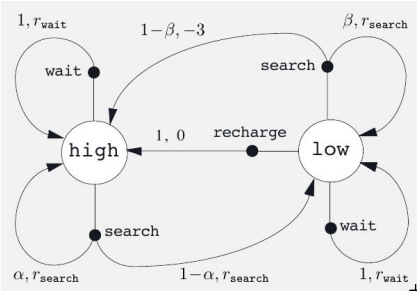

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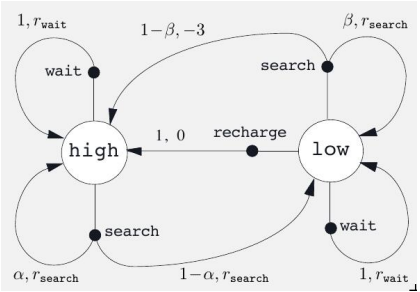
- α, β, probabilities associated with change of battery state while searching
- 1 unit of reward per can collected
- $r_{\text{search}} > r_{\text{wait}}$ — cans collected while searching, waiting
- Negative reward for requiring rescue (low to high while searching)



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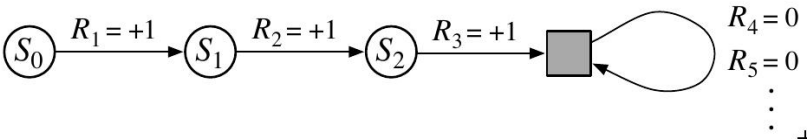
- Can make all episodes infinite by adding a self-loop with reward 0



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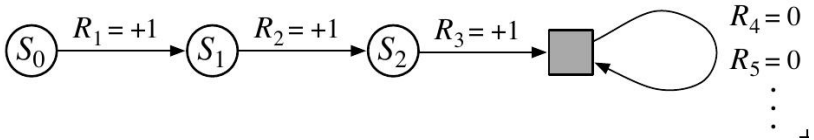

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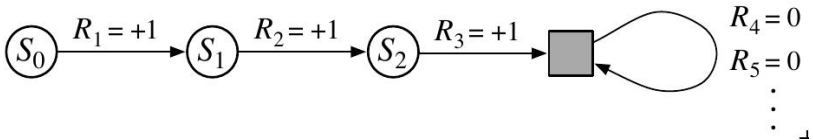

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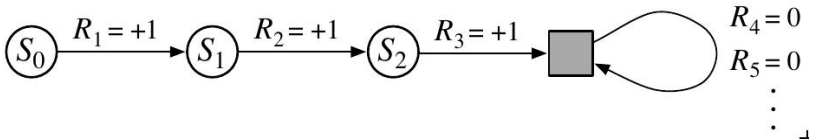

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- A policy π describes how the agent chooses actions at a state
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- Goal is to find an optimal policy, that maximizes state/action value at every state

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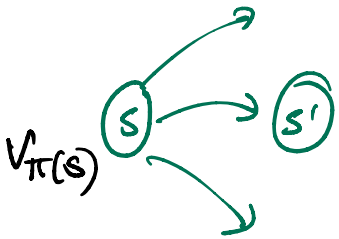

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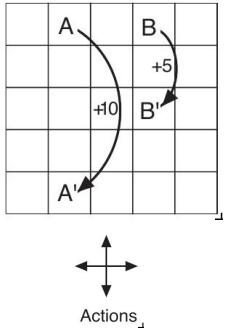
- Bellman equation relates state value at s to state values at successors of s
- Value function v_π is unique solution to the equation

Gridworld Example

- Actions in each cell are {N,S,E,W}, with usual interpretation



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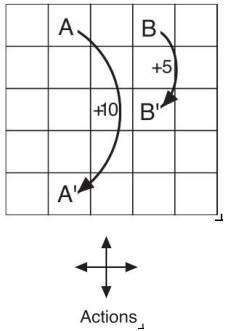

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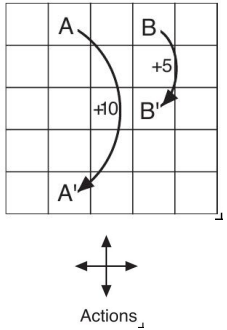

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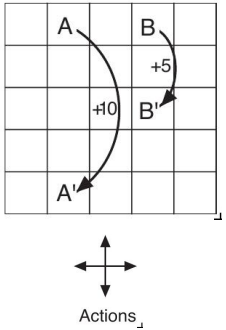

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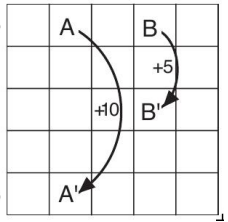

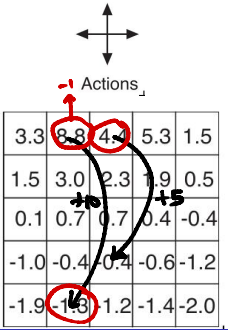

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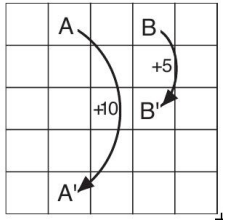
- Solving Bellman equations, we obtain v_π for each square
- Values at boundary are negative

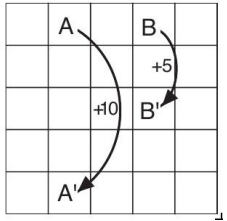


Actions

3.3	8.8	4.4	5.3	1.5
1.5	3.0	2.3	1.9	0.5
0.1	0.7	0.7	0.4	-0.4
-1.0	-0.4	-0.4	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

Gridworld Example

- Actions in each cell are {N,S,E,W}, with usual interpretation
- Reward is 0, except at boundaries
- Colliding with boundary — position unchanged, reward -1
- Special squares A and B — all four actions move as indicated, with rewards $+10$ and $+5$, respectively
- Policy π — choose each action with uniform probability 0.25
- Solving Bellman equations, we obtain v_π for each square
- Values at boundary are negative
- Value at A is less than 10 because next move takes agent to boundary square with negative value



Actions

3.3	8.8	4.4	5.3	1.5
1.5	3.0	2.3	1.9	0.5
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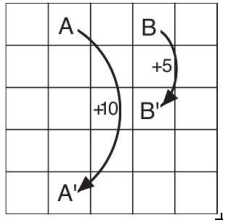

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- Instead, we will explore iterative methods to approximate v_*

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  - Stop when incremental change $\Delta = |v_\pi^{k+1} - v_\pi^k|$ is below threshold θ

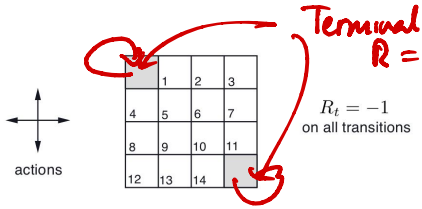
Policy evaluation

Iterative Policy Evaluation, for estimating $V \approx v_\pi$

Input π, the policy to be evaluated
Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
  $\Delta \leftarrow 0$
  Loop for each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s', r \,|\, s, a)\big[r + \gamma V(s')\big]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Policy evaluation example



v_k for the random policy

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

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- If $q_\pi(s,a) > v_\pi(s)$, modify π so that $\pi(s) = a$
- The new policy π' is strictly better

Policy improvement

Policy Improvement Theorem

For deterministic policies π, π':

- If $q_\pi(s, \pi'(s)) \geq v_\pi(s)$ for all s, then $\pi' \geq \pi$,
- If $\pi' \geq \pi$ and $q_\pi(s, \pi'(s)) > v_\pi(s)$ for some s, then $v_{\pi'}(s) > v_\pi(s)$.

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- The theorem extends to probabilistic policies also
- Provides a basis to iteratively improve the policy

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- Nested iteration — each policy evaluation is itself an iteration
  - Speed up by using v_{π_i} as initial state to compute $v_{\pi_{i+1}}$

Policy iteration

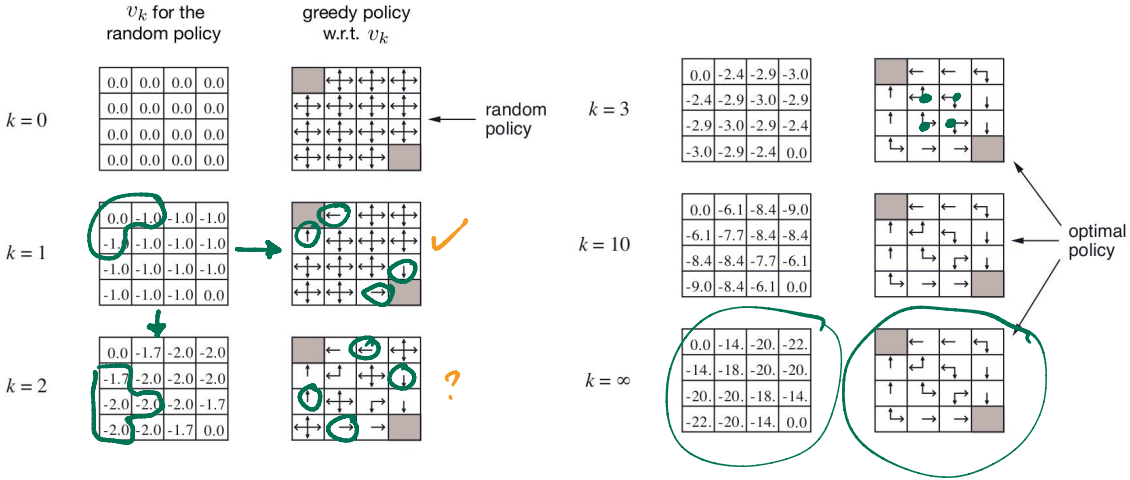
Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   $\Delta \leftarrow 0$
   Loop for each $s \in \mathcal{S}$:
   $v \leftarrow V(s)$
   $V(s) \leftarrow \sum_{s',r} p(s', r \mid s, \pi(s)) \big[r + \gamma V(s')\big]$
   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   old-action $\leftarrow \pi(s)$
   $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \mid s, a) \big[r + \gamma V(s')\big]$
   If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

Optimizing Policy Iteration



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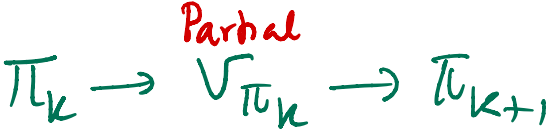

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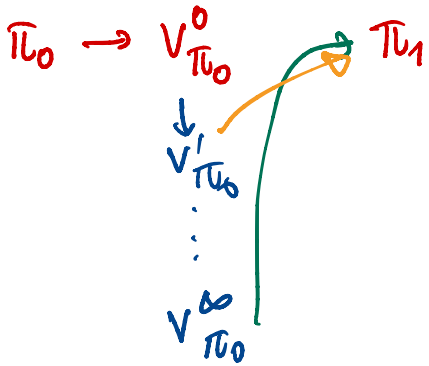

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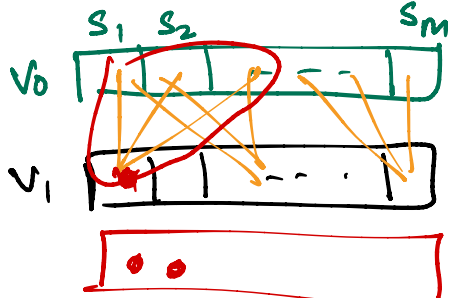
- Combine policy improvement and one step update at each state

Even $\quad V_\pi^0 \rightarrow U_\pi'$



Value iteration

- Policy iteration — policy evaluation requires a nested iteration
- A partial computation of v_{π_k} is sufficent to proceed towards π_*, v_*
- Even a single iteration in the computation of v_{π_k} will do
- Combine policy improvement and one step update at each state
- Value iteration

$$v_{\pi_{k+1}}(s,a) = \max_a \mathbb{E}[R_{t+1} + \gamma v_{\pi_k}(S_{t+1}) \mid S_t = s, A_t = a]$$

$$= \max_a \sum_{s',r} p(s', r \mid s, a) \left[r + \gamma v_{\pi_k}(s') \right]$$

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- Again, stop when incremental change $\Delta = |v_{\pi_{k+1}} - v_{\pi_k}|$ is below threshold θ

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  - Generalized policy iteration — simultaneously maintain and update approximations of π_* and v_*
- Asynchronous dynamic programming for large state spaces

Other approaches

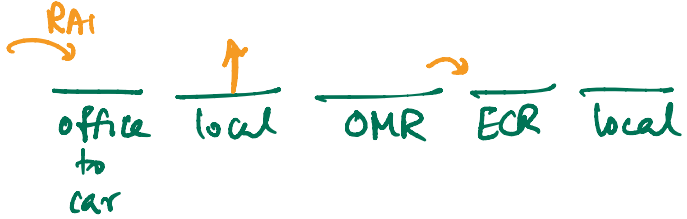
- Monte Carlo methods

Other approaches

- Monte Carlo methods
- Temporal-Difference learning — bootstrap estimates based on observations

Example

- Estimate travel time home = 45 minutes



Go

AlphaGo – used RL to beat Lee Sedol

Raw system

- Teach Go via RL
- Beat AlphaG

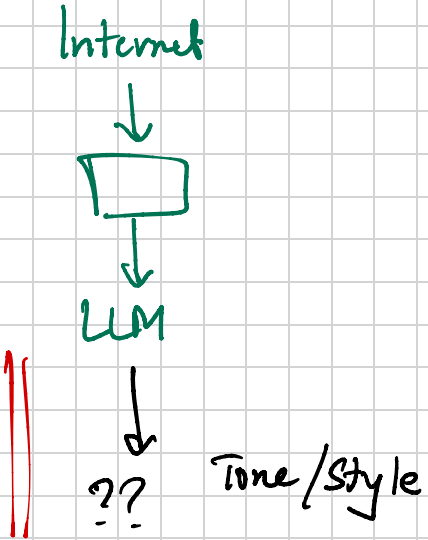
2019 - Yann Le Cun

How many cars will you crash?

LLMs

Pretraining

Fine tuning



Early fine tuning — Supervise

Sample Q & A

Today RLHF — Reinforcement learning with human feedback

Inflection → Pi