Automata for Real-time Systems

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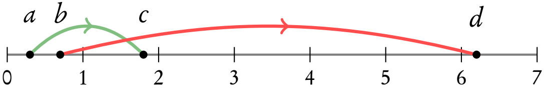
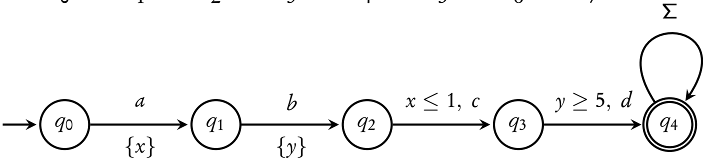
Lecture 2:

Timed languages and timed automata

$$L_5 := \{\ (\ abcd.\Sigma^*, \tau\ )\ |\ \tau_3 - \tau_1 \leq 2 \text{ and } \tau_4 - \tau_2 \geq 5\}$$

Interleaving distances





Exercise: Prove that L_5 cannot be accepted by a one-clock TA.

n interleavings $\Rightarrow$ need n clocks

$n + 1$ clocks more expressive than n clocks

$$\{\ (a^k, \tau) \mid \tau_{i+2} - \tau_i \leq 1 \text{ for all } i \leq k-2\}$$

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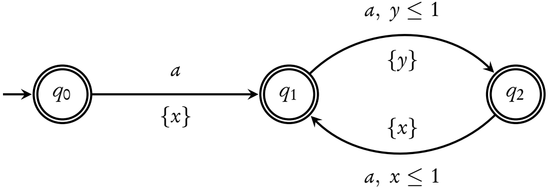

Timed automata

Runs

1 clock $<$ 2 clocks $< \ldots$

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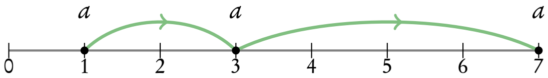

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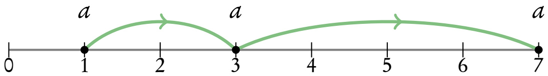

Claim: **No timed automaton** can accept L_6

Step 1: *Suppose* $L_6 = \mathcal{L}(A)$

Let c_{max} be the maximum constant appearing in a guard of A

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satisfy the same guards

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Step 3: $(a;\ \lceil c_{max} \rceil + 1) \in L_6$ and so A has an accepting run

$$(q_0, v_0) \xrightarrow{\delta\, =\, \lceil c_{max} \rceil + 1} (q_0, v_0 + \delta) \xrightarrow{a} (q_F, v_F)$$

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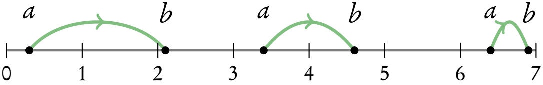
Hence $(a;\ \lceil c_{max} \rceil + 1.1) \in \mathcal{L}(A) \neq L_6$

Therefore **no timed automaton** can accept L_6 $\square$

$$L_7 = \{ \; (\, (ab)^k, \tau \;) \mid \tau_{2i+2} - \tau_{2i+1} < \tau_{2i} - \tau_{2i-1} \text{ for each } i \geq 1 \}$$

Converging *ab* distances



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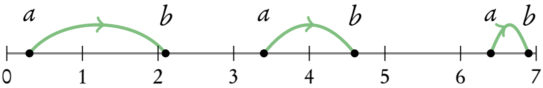
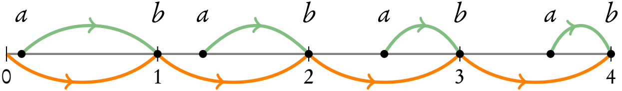


Exercise: Prove that **no timed automaton** can accept L_7

$$L_7 = \{ \; (\, (ab)^k, \tau \,) \mid \tau_{2i} = i \; \text{ and } \; \tau_{2i+2} - \tau_{2i+1} < \tau_{2i} - \tau_{2i-1} \}$$

Pivoted converging *ab* distances



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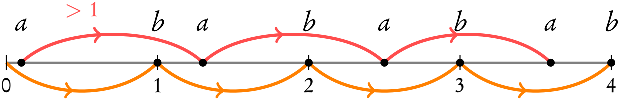

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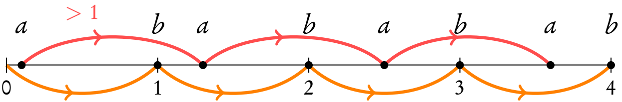

$$\tau_{2i+2} - \tau_{2i+1} < \tau_{2i} - \tau_{2i-1} \iff \tau_{2i+2} - \tau_{2i} < \tau_{2i+1} - \tau_{2i-1}$$

$$\iff 1 < \tau_{2i+1} - \tau_{2i-1}$$

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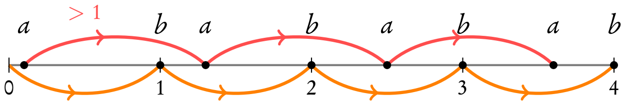

$$\tau_{2i+2} - \tau_{2i+1} < \tau_{2i} - \tau_{2i-1} \;\Leftrightarrow\; \tau_{2i+2} - \tau_{2i} < \tau_{2i+1} - \tau_{2i-1}$$

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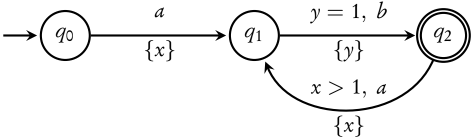


Timed automata

Runs

1 clock $<$ 2 clocks $<$ $\ldots$

Role of max constant

Timed automata

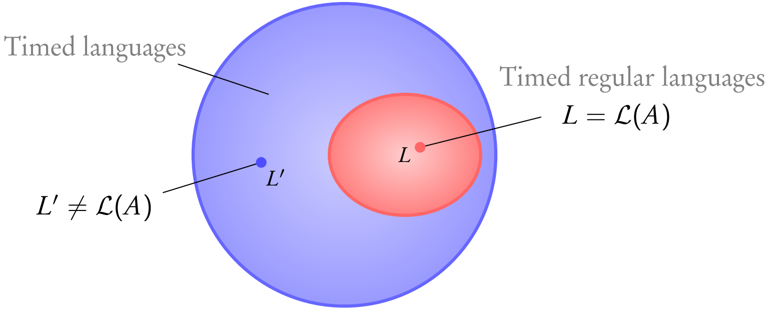
Runs

1 clock < 2 clocks < ...

Role of max constant

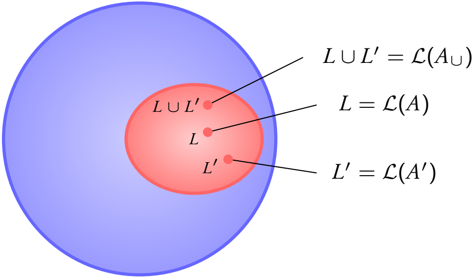
Timed regular lngs.

Timed regular languages



Definition

A timed language is called **timed regular** if it can be **accepted** by a timed automaton

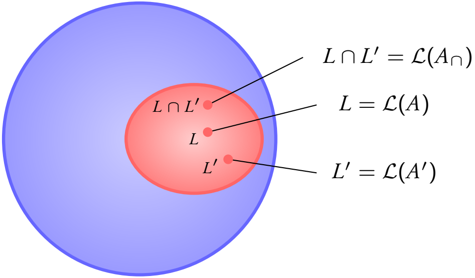


$$A = (Q, \Sigma, X, T, Q_0, F) \qquad A' = (Q', \Sigma, X', T', Q_0', F')$$

$$A_\cup = (\ Q \cup Q'\ ,\ \Sigma\ ,\ X \cup X'\ ,\ T \cup T'\ ,\ Q_0 \cup Q_0'\ ,\ F \cup F'\ )$$

$$\mathcal{L}(A) \cup \mathcal{L}(A') = \mathcal{L}(A_\cup)$$

Timed regular languages are **closed** under **union**



$$A = (Q, \Sigma, X, T, Q_0, F) \qquad A' = (Q', \Sigma, X', T', Q_0', F')$$

$$A_\cap = (\ Q \times Q'\ ,\ \Sigma\ ,\ X \cup X'\ ,\ T_\cap\ ,\ Q_0 \times Q_0'\ ,\ F \times F'\ )$$

$$T_\cap : \quad (q_1, q_1') \xrightarrow[R \cup R']{a,\ g \wedge g'} (q_2, q_2') \text{ if}$$

$$q_1 \xrightarrow[R]{a,\ g} q_2 \in T \quad \text{and} \quad q_1' \xrightarrow[R']{a,\ g'} q_2' \in T'$$

Timed regular languages are **closed** under **intersection**

L : a timed language over Σ

$$\text{Untime}(L) \equiv \{w \in \Sigma^* \mid \exists \tau.\ (w, \tau) \in L\}$$

Untiming construction

For **every timed** automaton A there is a **finite automaton** A_u s.t.

$$\text{Untime}(\ \mathcal{L}(A)\ ) = \mathcal{L}(A_u)$$

more about this later . . .

Complementation

$$\Sigma : \{a, b\}$$

$$L = \{ (w, \tau) \mid \text{there is an } a \text{ at some time } t \text{ and no action occurs at time } t+1 \}$$

$$\overline{L} = \{ (w, \tau) \mid \text{every } a \text{ has an action at a distance 1 from it} \}$$

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Claim: **No timed automaton** can accept $\overline{L}$

Decision problems for timed automata: A survey

Alur, Madhusudhan. *SFM'04: RT*

Step 1: $\overline{L} \;=\; \{\, (w, \tau) \mid$ every a has an action at a distance 1 from it $\}$

Suppose $\overline{L}$ is timed regular

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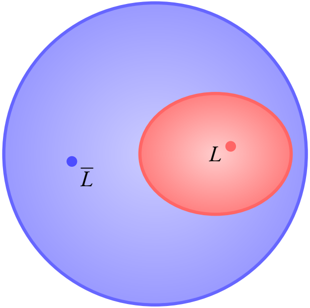
Therefore $\overline{L}$ **cannot be timed regular** $\square$



Timed regular languages are **not closed** under **complementation**

Timed automata

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Closure under $\cup$, $\cap$

Non-closure under complement

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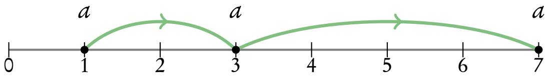
ε-transitions

$L_6 := \{\ (\ a^k, \tau\ )\ |\ \tau_i \text{ is some integer for each } i\}$



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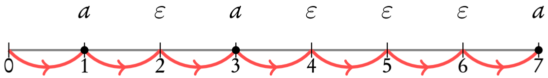

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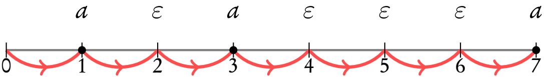

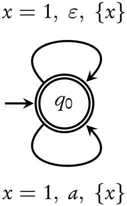

ε-transitions

ε-transitions **add expressive power** to timed automata.

Characterization of the expressive power of silent transitions in timed automata

Bérard, Diekert, Gastin, Petit. *Fundamenta Informaticae'98*

ε-transitions

ε-transitions **add expressive power** to timed automata. However, they add power **only** when a clock is **reset** in an ε-transition.

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Closure under $\cup, \cap$

Non-closure under complement

ε-transitions

More expressive

$\xrightarrow{\varepsilon}$ without reset $\equiv$ TA

Recall...

Huge system

↓

Higher-level description

↓ translation

Automaton $\mathcal{A}$

Property

↓

Higher-level description

↓ translation

Automaton $\mathcal{B}$

Model-Checker

$$\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})?$$

$$\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$$

iff

$$\mathcal{L}(\mathcal{A}) \cap \overline{\mathcal{L}(\mathcal{B})} = \emptyset$$

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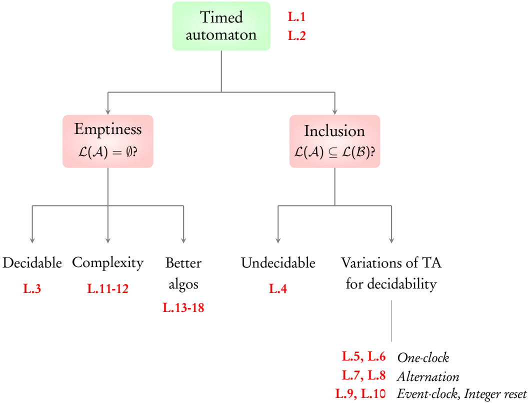
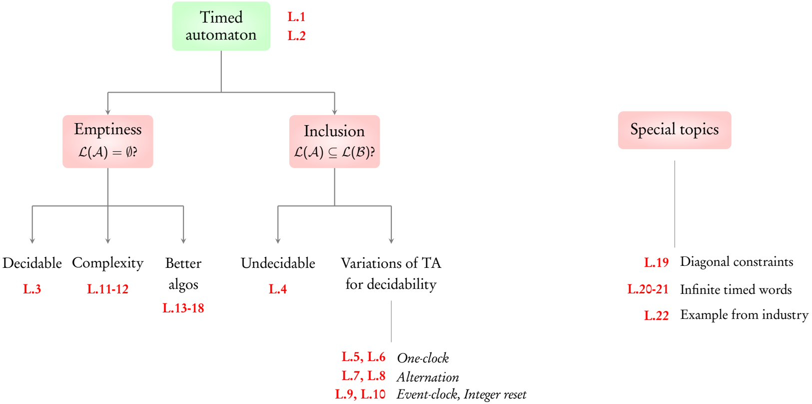
non-closure under complement $\Rightarrow$ the above **cannot be done** for TA!

Course plan


Timed automaton
L.1
L.2
Emptiness
$\mathcal{L}(\mathcal{A}) = \emptyset$?
Inclusion
$\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$?
Decidable
L.3
Complexity
L.11-12
Better algos
L.13-18
Undecidable
L.4
Variations of TA for decidability
L.5, L.6 *One-clock*
L.7, L.8 *Alternation*
L.9, L.10 *Event-clock, Integer reset*

Course plan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L.5, L.6 *One-clock*
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L.9, L.10 *Event-clock, Integer reset*
L.19 Diagonal constraints
L.20-21 Infinite timed words
L.22 Example from industry