Part 3.1 : From negative cycle to the test

Recall that:

$$Z \not\preccurlyeq_M Z' \iff \exists R.\ \ R \cap Z \neq \phi \ \text{ and } \ R \cap Z' = \phi$$

$$\iff \exists R.\ \ R \cap Z \neq \phi \ \text{ and } \ \exists x, y \in X \cup \{0\} \text{ s.t. } Z'_{xy} + R_{yx} < (\leq, 0)$$

(from previous proposition)

Now, we will show one side of the main theorem:

$\exists R.\ R \cap Z \neq \phi$ and $\exists x, y \in X \cup \{0\}$ s.t. $Z'_{xy} + R_{yx} < (\leq, 0)$

$\implies \exists x, y \in X \cup \{0\}$ s.t.

1. $Z_{x0} + (\leq, M_x) \geq (\leq, 0)$ and
2. $Z'_{xy} < Z_{xy}$ and
3. $Z'_{xy} + (<, -M_y) < Z_{x0}$

Proof: Let R be s.t. $R \cap Z \neq \phi$ and $R \cap Z' = \phi$.

To show (1).

Since $Z'_{xy} + R_{yx}$ is negative, $R_{yx} \neq (<, \infty)$

$\Rightarrow x \in \text{Bounded}(R)$ (recall that there are no incoming edges to unbounded clocks in R, that is, incoming edges to unbdd. clocks have $(<, \infty)$ weight)

This implies: $R_{0x} \leq (\leq, M_x)$

Adding Z_{x0} to both sides: $R_{0x} + Z_{x0} \leq (\leq, M_x) + Z_{x0}$

As $R \cap Z \neq \phi$, the cycle $R_{0x} + Z_{x0}$ in $\min(G_R, G_Z)$ is non-negative.

$$\therefore \quad (\leq, 0) \leq R_{0x} + Z_{x0}$$

This gives: $(\leq, 0) \leq (\leq, M_x) + Z_{x0}$ (condition 1).

<u>To show 2:</u>

We have $(\leq, 0) \leq Z_{xy} + R_{yx}$ as $R \cap Z \neq \phi$

$$Z'_{xy} + R_{yx} < (\leq, 0) \quad \text{by hypothesis.}$$

Adding: $Z'_{xy} < Z_{xy}$ (condition 2)

<u>To show 3:</u>

We will distinguish between two cases:

either $y \in \text{Bounded}(R)$ or $y \notin \text{Bounded}(R)$.

We start with the second case.

Case: $y \notin$ Bounded (R):

$$R_{yx} = (<, -M_y) + R_{0x} \quad \text{(by definition of } G_R\text{)}$$

[notice that x could as well be the 0 variable, in which case we have $R_{00} = (\leq, 0)$]

$$Z'_{xy} + R_{yx} < (\leq, 0) \quad \text{by hypothesis.}$$

$$\therefore \quad Z'_{xy} + (<, -M_y) + R_{0x} < (\leq, 0)$$

$$(\leq, 0) \leq R_{0x} + Z_{x0} \quad \text{as } R \cap Z \neq \phi$$

Adding: $\boxed{Z'_{xy} + (<, -M_y) < Z_{x0}}$ (condition 3)

Case: $y \in$ Bounded (R):

$$(<, -M_y) < R_{y0} \quad (\text{as } y \in \text{Bounded}(R))$$

$$R_{y0} \leq R_{yx} + R_{x0} \quad (\text{as } G_R \text{ is canonical})$$

$$Z'_{xy} + R_{yx} < (\leq, 0) \quad (\text{by hypothesis})$$

$$R_{x0} \leq Z_{x0} \quad (\text{as we will show later})$$

Adding: $Z'_{xy} + (<, -M_y) < Z_{x0}$ (condition 3)

Remains to show $R_{x0} \leq Z_{x0}$.

<u>To show:</u> $R_{xo} \leq Z_{xo}$

Suppose $R_{xo} = (\leq, -c)$ Then $R_{ox} = (\leq, c)$

We know: $$(\leq, 0) \leq Z_{xo} + R_{ox}$$

ie., $$(\leq, 0) \leq Z_{xo} + (\leq, c)$$

Adding $(\leq, -c)$ on both sides:

$$(\leq, -c) \leq Z_{xo}$$ [done]

Suppose: $R_{xo} = (<, -c)$. Then $R_{ox} = (<, c+1)$

We know: $$(\leq, 0) \leq Z_{xo} + R_{ox}$$

ie., $$(\leq, 0) \leq Z_{xo} + (<, c+1)$$

Suppose $Z_{xo} = (\triangleleft, c_{xo})$ where $\triangleleft \in \{<, \leq\}$

We have: $$(\leq, 0) \leq (\triangleleft, c_{xo}) + (<, c+1)$$

ie., $$(\leq, 0) \leq (<, c_{xo} + c + 1) \longrightarrow \circledast$$

Recall the ordering on weights: $(\leq, 0) \quad (<, 1) \quad (\leq, 1) \quad \cdots$

Since the RHS of $\circledast$ has strict inequality $<$, we also have:

$$(<, 1) \leq (<, c_{xo} + c + 1)$$

We have shown:

$$(<, 1) \leq (<, c_{x_0} + c + 1)$$

Adding $(<, -c-1)$ to both sides:

$$(<, -c) \leq (<, c_{x_0})$$

$$\therefore \quad (<, -c) \leq \underbrace{(<, c_{x_0}) < (\leq, c_{x_0})}_{\text{by ordering on weights}}$$

$$\Rightarrow \quad (<, -c) \leq Z_{x_0} \quad \text{no matter what } \triangleleft \text{ is.}$$

(recall $Z_{x_0} = (\triangleleft, c_{x_0})$)

i.e., $R_{x_0} \leq Z_{x_0}$.

Alternate elegant proof given by Niranjan for this case:

Case: $y \in \text{Bounded}(R)$

$$R_{y0} \leq R_{yx} + R_{x0} \qquad \text{(by canonicity of } G_R)$$

$$R_{0x} + R_{x0} \leq (<, 1) \qquad \text{(by defininition of } G_R)$$

$$Z'_{xy} + R_{yx} < (\leq, 0)$$

Adding all three:

$$Z'_{xy} + R_{0x} + R_{y0} < (<, 1)$$

$$\Rightarrow \quad Z'_{xy} + R_{0x} + R_{y0} \leq (\leq, 0) \qquad \text{[due to ordering on weights]} \quad \text{①}$$

$$(\leq, 0) \leq R_{0x} + Z_{x0} \qquad [\text{as } R \cap Z \neq \phi] \quad \text{②}$$

$$(<, -M_y) < R_{y0} \qquad [\text{as } y \in \text{Bounded}(R)] \quad \text{③}$$

Adding ①, ②, ③ gives:

$$Z'_{xy} + (<, -M_y) < Z_{x0} \qquad !!$$

Part 3.2: From the test to a negative cycle.

Given: $\exists x, y \in X \cup \{0\}$ s.t.

- 1. $Z_{x0} + (\leq, M_x) \geq (\leq, 0)$ and
- 2. $Z'_{xy} < Z_{xy}$ and
- 3. $Z'_{xy} + (<, -M_y) < Z_{x0}$

To show:

$\exists R.\ R \cap Z \neq \emptyset$ and $\exists x, y \in X \cup \{0\}$ s.t.

$Z'_{xy} + R_{yx} < (\leq, 0)$

Proof: We will do a case distinction depending on the inequality present in Z_{x0} and Z_{xy}.

Case 1: $Z_{x0} = (\leq, -c)$, $Z_{xy} = (\leq, e)$

Case 2: $Z_{x0} = (<, -c)$, $Z_{xy} = (<, e)$

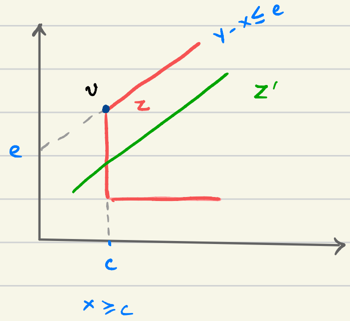
Case 3: $Z_{x0} = (<, -c)$, $Z_{xy} = (\leq, e)$

Case 4: $Z_{x0} = (\leq, -c)$, $Z_{xy} = (<, e)$

We will describe Cases 1 and 2. The other two are left as an exercise.

Moreover, we will assume that $x, y \in X$, that is, neither x nor y is 0. The case where one of them is 0 can be argued similarly.

Case 1: $Z_{x0} = (\leq, c)$ and $Z_{xy} = (\leq, e)$



Consider the point v marked with the blue dot.

$v(x) = c$, $v(y) = e$

We claim that the region containing v witnesses a negative cycle with Z'.

Let R be the region containing v.

- Claim 1: $x \in$ Bounded (R).

By ①, $Z_{x0} + (\leq, M_x) \geq (\leq, 0)$

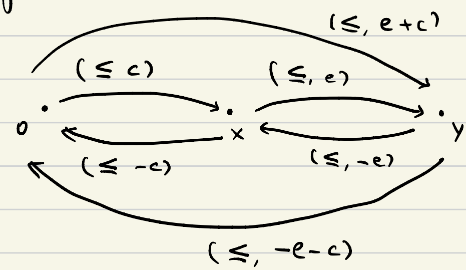
$\therefore (\leq, -c) + (\leq, M_x) \geq (\leq, 0)$

$\therefore (\leq, -c + M_x) \geq 0$

$\Rightarrow \quad c \leq M_x.$ $\square$

- y can either be bounded or not.

Suppose y is bounded. Then R will be:

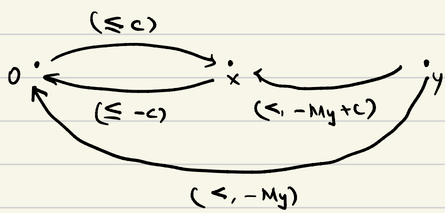


$R_{yx} = (\leq, -e)$

We know: $\quad Z'_{xy} < Z_{xy} = (\leq, e)$

$\therefore \quad Z'_{xy} + (\leq, -e) < (\leq, 0) \qquad$ (done).

Suppose y is not bounded. Then R will be:



$R_{yx} = (<, -M_y + c)$

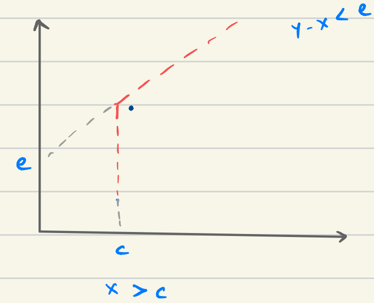
From 3: $\quad Z'_{xy} + (<, -M_y) < Z_{x0}$

ie., $\quad Z'_{xy} + (<, -M_y) < (\leq, c)$

Adding $(\leq, -c)$: $\quad Z'_{xy} + (<, -M_y + c) < (\leq, 0)$

$\Rightarrow \quad Z'_{xy} + R_{yx} < (\leq, 0)$

Case 2: $\quad Z_{x0} = (<, -c)$, $\quad Z_{xy} = (<, e)$



Consider the point

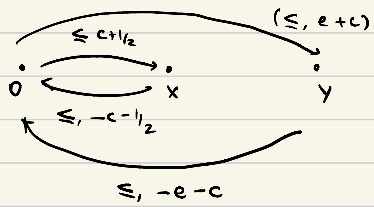
$x = c + 1/2$

$y = e + c$

- **Claim:** There exists $v \in Z$ s.t. $\quad v(x) = c + 1/2$

$v(y) = e + c$

To prove this, we will show that $\min(G_1, G_2)$ has no negative cycles where G_1 is:



It is sufficient to show the following cycles are non-negative

i) $\quad (\leq, c + 1/2) + Z_{x0}$ $\qquad$ Why? $\quad$ (Exercise)

ii) $\quad Z_{0x} + (\leq, -c - 1/2)$

iii) $\quad (\leq, e + c) + Z_{y0}$

iv) $\quad Z_{0y} + (\leq, -e - c)$ $\quad$ — $\quad$ (Exercise)

v) $\quad (\leq, c + 1/2) + Z_{xy} + (\leq, -e - c)$

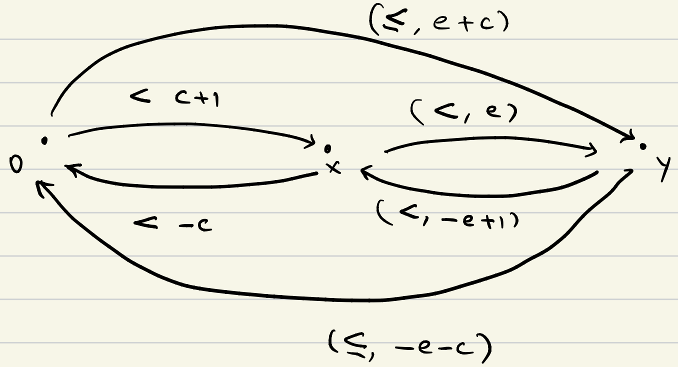
vi) $\quad (\leq, e + c) + Z_{yx} + (\leq, -c - 1/2)$

Now, we will show that the region containing v will satisfy $Z'_{xy} + R_{yx} < (\leq, 0)$.

As in case 1, we can show that x is bounded.

<u>Suppose</u> <u>y is bounded</u>: ie., $e + c \leq M_y$.

The region R will look like:



(Recall that both e and c are integers, and hence this is the region).

$$R_{yx} = (<, -e+1)$$

We know: $Z'_{xy} < Z_{xy} = (<, e)$ (from ②)

$$\therefore \quad Z'_{xy} \leq (\leq, e-1)$$

Adding R_{yx}:

$$Z'_{xy} + (<, -e+1) \leq (<, 0)$$

$$\Rightarrow \quad Z'_{xy} + R_{yx} < (\leq, 0) \quad \text{[negative cycle]}$$