Part 3: Proof of the final test

We have already seen that:

$z \not\preccurlyeq_M z'$ if $\exists$ an M-region R s.t.

$$R \cap z \neq \phi \quad \text{and} \quad R \cap z' = \phi$$

Our first step would be to understand when $R \cap z'$ is empty.

When is $R \cap z'$ empty?

R is some arbitrary M-region and Z' is a non-empty zone.

Let us first look at the canonical distance graph G_R representing R.

Define

$$\boxed{\text{Bounded}(R) = \{0\} \cup \{x \in X \mid \text{every valuation in } R \text{ satisfies } x \leq M_x\}}$$

Notice that we have added $\{0\}$ to Bounded (R), as this will help avoid separate arguments for the 0-variable.

Here is a distance graph for R:

- $\forall x \in \text{Bounded}(R) \cap X$ s.t. $x = c$ for some $c \leq M_x$

$$\begin{tikzcd} 0 \arrow[rr, bend left, "(\leq c)"] & & x \arrow[ll, bend left, "(\leq -c)"] \end{tikzcd}$$

- $\forall x \in \text{Bounded}(R) \cap X$ s.t. $c < x < c+1$ for some $c < M_x$

$$\begin{tikzcd} 0 \arrow[rr, bend left, "(<, c+1)"] & & x \arrow[ll, bend left, "(<, -c)"] \end{tikzcd}$$

- $\forall x_1, x_2 \in \text{Bounded}(R)$ s.t. $c_1 < x_1 < c_1+1$ &
$c_2 < x_2 < c_2+1$

one of the following combinations

$$\begin{tikzcd} (c_1, c_2+1) \arrow[r, dash] \arrow[d, dash] & (c_1+1, c_2+1) \arrow[d, dash] \\ (c_1, c_2) \arrow[r, dash] & (c_1+1, c_2) \end{tikzcd}$$

$$\begin{tikzcd} x_1 \arrow[rr, bend left, "(<, c_2+1-c_1)"] & & x_2 \arrow[ll, bend left, "(<, c_1-c_2)"] \end{tikzcd}$$

$$\begin{tikzcd} x_1 \arrow[rr, bend left, "(\leq, c_2-c_1)"] & & x_2 \arrow[ll, bend left, "(\leq, c_1-c_2)"] \end{tikzcd}$$

$$\begin{tikzcd} x_1 \arrow[rr, bend left, "(<, c_2-c_1)"] & & x_2 \arrow[ll, bend left, "(<, c_1+1-c_2)"] \end{tikzcd}$$

- $\forall x \notin \text{Bounded}(R)$

$$\begin{tikzcd} 0 & & y \arrow[ll, bend left, "(<, -M_y)"] \end{tikzcd}$$

This graph may not be canonical.

The canonical version of the graph just defined will have the following additional edges:

- for all $x_1, x_2 \in$ Bounded $(\mathbb{R})$ s.t.

$$\begin{tikzcd} 0 \arrow[r, bend left, "\leq c_1"] & x_1 \arrow[l, bend left, "\leq -c_1"] \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r, bend left, "\leq c_2"] & x_2 \arrow[l, bend left, "\leq -c_2"] \end{tikzcd}$$

we have:

$$\begin{tikzcd} x_1 \arrow[r, bend left, "{\leq,\ c_2 - c_1}"] & x_2 \arrow[l, bend left, "{\leq,\ c_1 - c_2}"] \end{tikzcd}$$

- for $y \notin$ Bounded $(\mathbb{R})$, $x \in$ Bounded $(\mathbb{R}) \setminus \{0\}$

$$\begin{tikzcd} x & y \arrow[l, bend left, "{(\lhd,\ w)}"] \end{tikzcd}$$

where $(\lhd, w) = (<, -M_y + c)$, if weight of $0 \to x$ is $(\lhd_1, c)$

(wt. of $y \to 0 \to x$)

Let G_R be the canonical distance graph of R, as defined in the previous two pages.

Here are two useful properties of G_R that we will use crucially.

Lemma: Let $x_1, x_2 \in$ Bounded (R). Then weight of the cycle

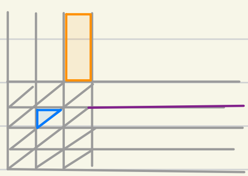
$$\begin{tikzcd} x_1 \arrow[r, bend left] & x_2 \arrow[l, bend left] \end{tikzcd}$$

in G_R is either $(\leq, 0)$ or $(<, 1)$.

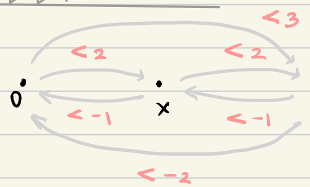
Lemma: G_R has no incoming edges to variables $y \notin$ Bounded (R)

Proof of the above two lemmas simply follows by definition.

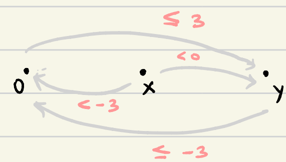
Example:



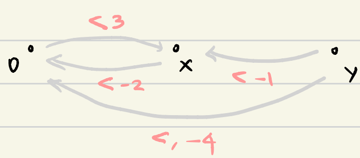
Blue region:



Violet region:



Orange region:



Proposition: Let R be an M-region, and Z' a non-empty zone.

$R \cap Z' = \phi$ iff there exist two variables $x, y \in X \cup \{0\}$ s.t.

$$\boxed{Z'_{xy} + R_{yx} < (\leq, 0)}$$

where Z'_{xy} is the weight of $x \longrightarrow y$ in the canonical distance graph $G_{Z'}$ of Z'

R_{yx} is the weight of $y \longrightarrow x$ in the canonical distance graph G_R of R

Proof: Consider $\min(G_R, G_{Z'})$. We know that

$R \cap Z'$ is empty iff $\min(G_R, G_{Z'})$ has a negative cycle.

The proposition claims that we can find a small negative cycle:

$$\begin{tikzcd} x \arrow[r, bend left, "Z'_{xy}"] & y \arrow[l, bend left, "R_{yx}"] \end{tikzcd}$$

in $\min(G_R, G_{Z'})$ where weight of $x \longrightarrow y$ comes from $G_{Z'}$ and weight of $y \longrightarrow x$ comes from G_R.

Right-to-left direction: Suppose the rhs of the proposition is true. This means there is a negative cycle in $\min(G_R, G_{Z'})$. Hence $R \cap Z' = \phi$.

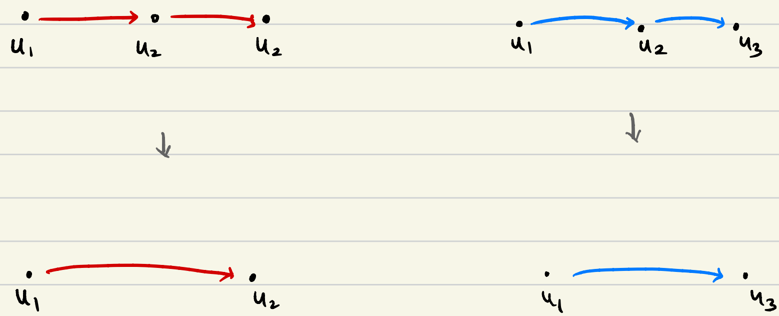
Left - to - right direction:

Assume $R \cap z' = \phi$. Then $\min(G_R, G_{z'})$ has a negative cycle.

— Let N be a negative cycle in $\min(G_R, G_{z'})$. We will reduce N to the required form step by step.

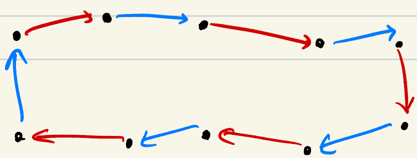
Convention: We will colour edges coming from G_R with red and edges coming from $G_{z'}$ with blue.

Step 1: Since $G_R, G_{z'}$ are canonical, any consecutive occurrences of red or blue edges can be replaced by a single direct edge from source to target, that has smaller or equal weight.



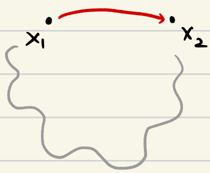
Therefore we can assume that red & blue edges alternate in N.

N:



Step 2: We will now reduce N so that it contains at most 2 variables $x_1, x_2 \in$ Bounded (R). Moreover x_1 and x_2 are connected by a direct red edge.

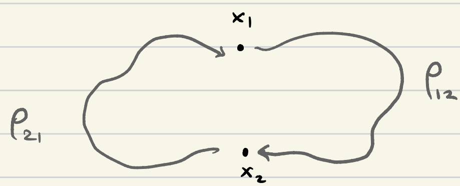
To show, N looks like:



gray part has no variables from Bounded (R).

Proof:

Suppose N contains two variables $x_1, x_2 \in$ Bounded (R):



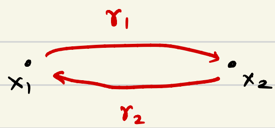
Let P_{12} be the sequence of edges (possibly a single edge) from x_1 to x_2 in N.

P_{21} be the sequence of edges (possibly a single one) from x_2 to x_1 in N.

Let weight of P_{12} be w_{12}

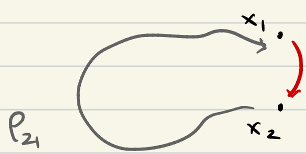
and weight of P_{21} be w_{21}.

In G_R, we have



with $r_1 + r_2 = (\leq, 0)$ or
$r_1 + r_2 = (<, 1)$ (as reworded in a previous lemma)

Suppose $r_1 \leq w_1$, then



is a negative cycle.

Else: $w_1 < r_1$

$\therefore \; w_1 + r_2 < r_1 + r_2$

When $r_1 + r_2 = (\leq, 0)$, we get $w_1 + r_2 < (\leq, 0)$
When $r_1 + r_2 = (<, 1)$, we get $w_1 + r_2 < (<, 1)$

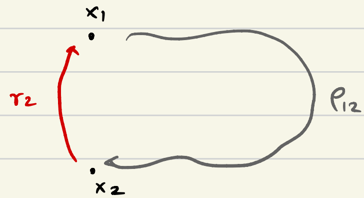
$\Rightarrow \; w_1 + r_2 \leq (\leq, 0)$

But, in this case r_2 is of the form $(<, e)$
strict inequality

$\therefore \; w_1 + r_2$ is of the form $(<, f)$

$\Rightarrow \; w_1 + r_2 < (\leq, 0)$

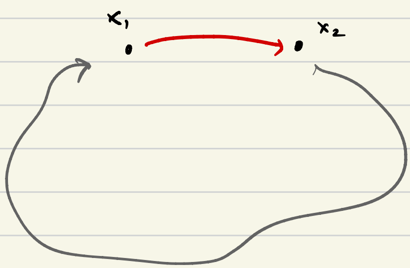
$\therefore$



is a negative cycle.

Either way, we have seen that we get a negative cycle where x_1, x_2 are connected by a direct edge.

If there are more bounded variables (say in ρ_{21} or ρ_{12}), we can repeat this argument, each time eliminating a bounded variable, to finally get a negative cycle that looks like:



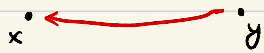
where $x_1, x_2 \in$ Bounded (R)

and the rest of the variables are not in Bounded (R).

Step 3: We will now show that there can be at most one "unbounded" variable in N.

Suppose $y \notin$ Bounded (R) and y is present in N.

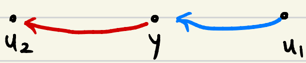
(i) We have seen that in G_R, the only edges involving y are of the form:



where $x \in$ Bounded (R).

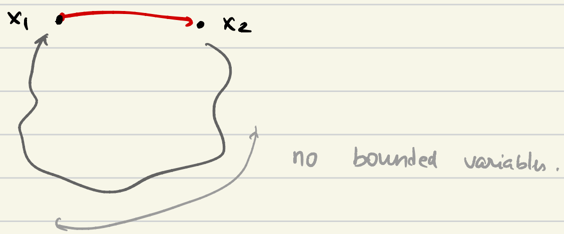
(ii) From Step 1, red and blue edges alternate in N.

From (i) and (ii), if y is present in N, then

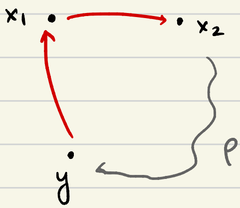


the incoming edge to y is blue, and outgoing edge is red going to a bounded variable.

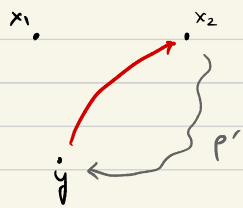
Recall that, from Step 2, N looks like:



If y is in the grey part, then the outgoing red edge from y has to go to either x_1 or x_2.



or

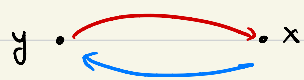


But as there are no consecutive red edges, it can only be the second option.

The grey part ρ' does not contain bounded variables (step 2)

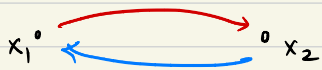
If ρ' contains an unbounded variable y' (other than y), then we can apply the same argument as we did with y to get a direct edge y' to x_2. This will eliminate y from N.

$\therefore$ Wlog, N looks like:

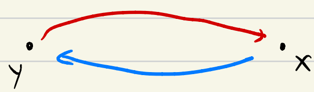


In summary,

either N has no unbounded variables, in which case Step 2 gives a negative cycle:



or N contains at least one unbounded variable, in which case Step 2 and Step 3 together give a negative cycle:



This proves the proposition.

Remark:

The above proposition shows that:

$$R \cap Z = \phi \quad \text{iff} \quad \exists\, x, y \in X \cup \{0\} \quad \text{s.t.}$$

$$\mathrm{Proj}_{xy}(R) \cap \mathrm{Proj}_{xy}(Z') = \phi$$

where $\mathrm{Proj}_{xy}(S)$ denotes the projection of S in the xy plane.