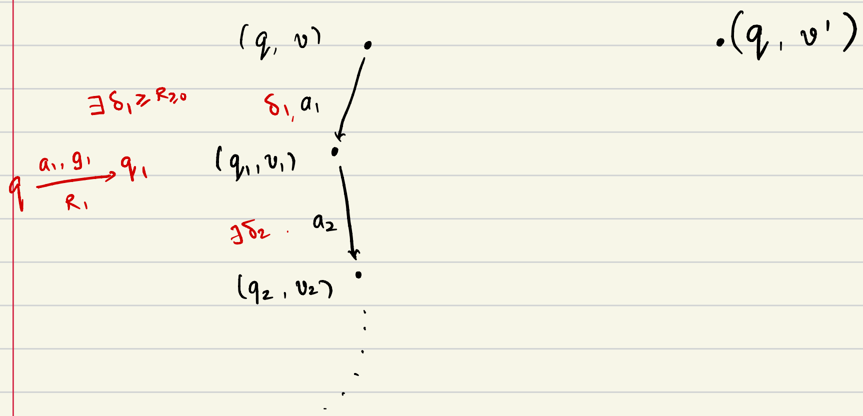
TIMED AUTOMATA

LECTURE 3

Previous lecture:

- Given a timed automaton $\mathcal{A}$, we constructed an infinite state automaton for $Untime(\mathcal{L}(\mathcal{A}))$



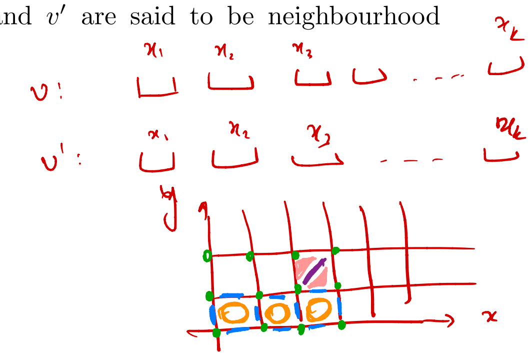
Merging configurations:

In the lecture we have seen neighbourhood equivalence when there are only two clocks $\{x, y\}$. This can be extended to the general case with multiple clocks by considering clocks pairwise. Here is the definition.

Definition 1 (Neighbourhood equivalence) Two valuations v and v' are said to be neighbourhood equivalent, written as $v \simeq_{\mathsf{nbd}} v'$ if:

1. $\lfloor v(x) \rfloor = \lfloor v'(x) \rfloor$ for all clocks x,
2. $\{v(x)\} = 0$ iff $\{v'(x)\} = 0$ for all clocks x
3. for every pair of clocks x, y:
   (a) $\{v(x)\} < \{v(y)\} \Leftrightarrow \{v'(x)\} < \{v'(y)\}$
   (b) $\{v(x)\} = \{v(y)\} \Leftrightarrow \{v'(x)\} = \{v'(y)\}$

Each equivalence class of $\simeq_{\mathsf{nbd}}$ will be called a *neighbourhood*.



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1. Group the following valuations over two clocks into neighbourhoods.

[$v_1 \simeq_{nbd} v_4 \simeq_{nbd} v_7$]
(v_3)
($v_2 \simeq_{nbd} v_6$)
(v_5)

$v_1 := (3.4,\ 7.2)$ 3, 7 $\{y\} < \{x\}$
$v_2 := (2.1,\ 10.0)$ 2 10 $\{y\} = 0$, $\{x\} \neq 0$
$v_3 := (3.3,\ 7.4)$ 3 7 $\{x\} < \{y\}$
$v_4 := (3.7,\ 7.2)$ 3 7 $\{y\} < \{x\}$
$v_5 := (2.1,\ 10.1)$ 2 10 $\{x\} = \{y\} \neq 0$
$v_6 := (2.7,\ 10.0)$ 2 10 $\{y\} = 0$, $\{x\} > 0$
$v_7 := (3.9,\ 7.4)$ 3 7 $\{y\} < \{x\}$

2. Below are combinations of v, δ and v' with $v \simeq_{\mathsf{nbd}} v'$. For each of them, find a $\delta' \geq 0$ such that $v + \delta \simeq_{\mathsf{nbd}} v' + \delta'$.

$$v = (2.2, 3.7) \qquad \delta = 0.3 \qquad v' = (2.1, 3.9)$$
$$v = (5.0, 8.2) \qquad \delta = 3.8 \qquad v' = (5.0, 8.9)$$
$$v = (3.2, 2.1) \qquad \delta = 0.9 \qquad v' = (3.7, 2.2)$$
$$v = (3.2, 2.1) \qquad \delta = 2.9 \qquad v' = (3.7, 2.2)$$

Recall that $v + \delta$ is the valuation obtained by adding δ to each coordinate of v.

3. Group the following valuations over five clocks into neighbourhoods.

$v_1 \simeq_{nbd} v_4$
$v_2 \simeq_{nbd} v_6$
v_3
v_5

x_1 x_2 x_3 x_4 x_5
$v_1 := (7.4,\ 2.1,\ 8.7,\ 5.4,\ 7.0)$ 7, 2, 8, 5, 7 $\{x_5\}$ $x_2 < x_1 = x_4 < x_3$
$v_2 := (3.4,\ 2.0,\ 8.5,\ 10.0,\ 7.1)$ 3, 2, 8, 10, 7 $\{x_2, x_4\}$ $x_5 < x_1 < x_3$
$v_3 := (7.3,\ 2.2,\ 8.8,\ 5.2,\ 7.0)$ 7, 2, 8, 5, 7 $\{x_5\}$ $x_2 = x_4 < x_1 < x_3$
$v_4 := (7.5,\ 2.1,\ 8.9,\ 5.5,\ 7.0)$ 7, 2, 8, 5, 7 $\{x_5\}$ $x_2 < x_1 = x_4 < x_3$
$v_5 := (3.2,\ 2.0,\ 8.8,\ 10.0,\ 7.5)$ 3, 2, 8, 10, 7 $\{x_2, x_4\}$ $x_1 < x_5 < x_2$
$v_6 := (3.3,\ 2.0,\ 8.4,\ 10.0,\ 7.2)$ 3, 2, 8, 10, 7 $\{x_2, x_4\}$ $x_5 < x_1 < x_3$

4. Below are combinations of v, δ and v' with $v \simeq_{\mathsf{nbd}} v'$. For each of them, find a $\delta' \geq 0$ such that $v + \delta \simeq_{\mathsf{nbd}} v' + \delta'$.

$$v = (1.1,\ 2.0,\ 3.0,\ 4.0,\ 5.9) \qquad \delta = 3.4 \qquad v' = (1.5,\ 2.0,\ 3.0,\ 4.0,\ 5.6)$$
$$v = (10.2,\ 4.8,\ 19.1,\ 2.0,\ 8.5) \qquad \delta = 5.4 \qquad v' = (10.1,\ 4.9,\ 19.05,\ 2.0,\ 8.7)$$

5. Now we look at the general case. Let v, v' be valuations such that $v \simeq_{\mathsf{nbd}} v'$. Show that for every $\delta \geq 0$, there exists a $\delta' \geq 0$ such that $v + \delta \simeq_{\mathsf{nbd}} v' + \delta'$

i) $v = (2.2, 3.7)$	$\delta = 0.3$	$v' = (2.1, 3.9)$
ii) $v = (5.0, 8.2)$	$\delta = 3.8$	$v' = (5.0, 8.9)$
iii) $v = (3.2, 2.1)$	$\delta = 0.9$	$v' = (3.7, 2.2)$
iv) $v = (3.2, 2.1)$	$\delta = 2.9$	$v' = (3.7, 2.2)$

Find δ' s.t. $v + \delta \simeq_{nbd} v' + \delta'$.

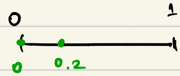
i) $v : (2.2, \; 3.7)$ $\quad \delta = 0.3$

$v + \delta : (2.5, \; 4.0)$

$v' : (2.1, \; 3.9)$ $\quad \delta' = 0.1$

$v' + \delta' = (2.2, \; 4.0)$

ii) $v = (5.0, \; 8.2)$ $\quad \delta = 3.8$

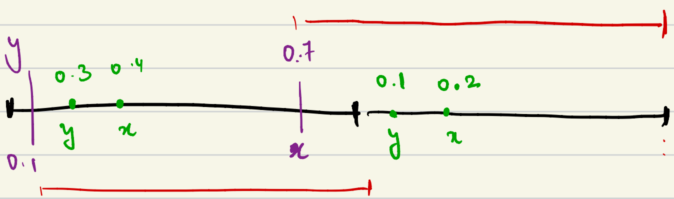


$v \; (5.0, \; 8.2) \xrightarrow{\lfloor\delta\rfloor = 3.0} (8.0, \; 11.2) \xrightarrow{\{\delta\} = 0.8} (8.8, \; 12.0)$

$v' : (5.0, \; 8.9) \xrightarrow{3.0} (8.0, \; 11.9) \xrightarrow{0.1} (8.1, \; 12.0)$

v) $v = (3.4, \; 2.3)$ $\quad \delta = 0.8$ $\quad v' : (3.7, \; 2.1)$

$v + \delta : (4.2, \; 3.1)$ $\quad \delta' < 1.3$



$\delta' > 0.9$

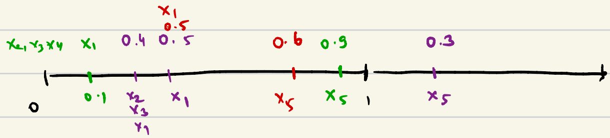
$v = (1.1,\ 2.0,\ 3.0,\ 4.0,\ 5.9)$ $\quad \delta = 3.4$ $\quad v' = (1.5,\ 2.0,\ 3.0,\ 4.0,\ 5.6)$

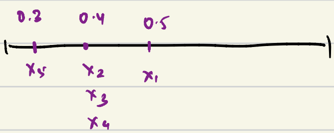
$v = (10.2,\ 4.8,\ 19.1,\ 2.0,\ 8.5)$ $\quad \delta = 5.4$ $\quad v' = (10.1,\ 4.9,\ 19.05,\ 2.0,\ 8.7)$

$$v = (\overset{x_1}{1.1},\ \overset{x_2}{2.0},\ \overset{x_3}{3.0},\ \overset{x_4}{4.0},\ \overset{x_5}{5.9}) \qquad \delta = 3.4 \qquad v' = (1.5, 2.0, 3.0, 4.0, 5.6)$$

$$\Big\downarrow +3 \quad (L\varepsilon) \qquad\qquad \Big\downarrow 3$$

$$(4.1,\ 5.0,\ 6.0,\ 7.0,\ 8.9) \qquad (4.5,\ 5.0,\ 6.0,\ 7.0,\ 8.6)$$





$$3.4 < \delta' < 3.5$$

GOALS OF TODAY'S LECTURE:

1. Neighbourhood equivalence $\longrightarrow$ aut. $NBD(\mathcal{A})$
   - accepts $Untime(\mathcal{L}(\mathcal{A}))$
   - infinitely many states

2. Region equivalence $\longrightarrow$ aut. $Reg(\mathcal{A})$
   - accepts $Untime(\mathcal{L}(\mathcal{A}))$
   - finitely many states

THREE OBSERVATIONS ABOUT

Neighbourhood equivalence

$\simeq_{nbd}$

Lemma 1: Let $v \simeq_{nbd} v'$.

For every $\delta \geq 0$, there exists $\delta' \geq 0$ s.t. $v + \delta \simeq_{nbd} v' + \delta'$

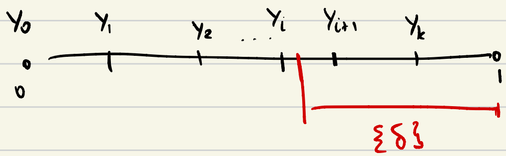
Proof:

$$\begin{tikzcd} v \arrow[r, phantom, "\simeq"] \arrow[d, "\delta \geq 0"'] & v' \arrow[d, "\exists \delta' \geq 0"] \\ v + \delta \arrow[r, phantom, "\simeq"] & v' + \delta' \end{tikzcd}$$

Choose $\lfloor \delta' \rfloor = \lfloor \delta \rfloor$. It can be shown that $v + \lfloor \delta \rfloor \simeq_{nbd} v' + \lfloor \delta' \rfloor$

Now we need to find $\{\delta'\}$

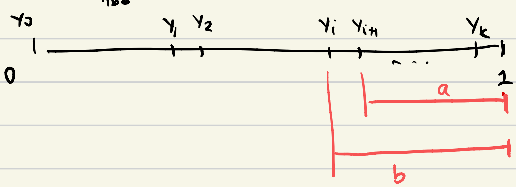
Suppose the ordering of fractional parts in v (and $v + \lfloor \delta \rfloor$) is as follows:



By adding $\{\delta\}$, clocks in $y_{i+1}, \ldots, y_k$ go to the next integer
$y_0, \ldots y_i$ stay within same integer.

We need to mimick this in v'.

Since $v' \simeq_{nbd} v'$:



Choose $\{\delta'\}$ in the open interval (a, b)

Lemma 2: Let $v \simeq_{nbd} v'$.

For every guard φ : v satisfies φ iff v' satisfies φ.

(constants are natural nos in guards)

Guards: conjunction of: $x \sim c$ | $\sim \in \{<, \leq, =, >, \geq\}$, $c \in \mathbb{N}$

$x > 1 \wedge y = 2 \wedge x > 4$

Proof: Since $v \simeq_{nbd} v'$

~~we have~~ 1) $\lfloor v(x) \rfloor = \lfloor v'(x) \rfloor \quad \forall x$

1.2 1.3

1.. $\underline{1.0}$ $x < 1$

2) $\{v(x)\} = 0$ iff $\{v'(x)\} = 0$

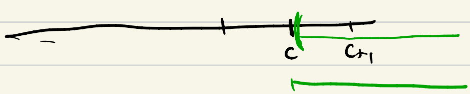
One can show for each atomic constraint:

$v \models x < c$ iff $v' \models x < c$

$v \models x \leq c$ $v' \models x \leq c$

$v \models x \geq c$ iff $v' \models x \geq c$

$v \models x > c$ iff $v' \models x > c$



$\lfloor v(x) \rfloor$ can be c

$\{v(x)\} \neq 0$ if $\lfloor v(x) \rfloor = c$

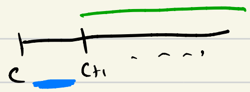
$v \simeq_{nbd} v'$

For eg:

$v \models x > c$ iff $v' \models x > c$

Proof: $v \models x > c$

This implies either $\lfloor v(x) \rfloor \geq c+1$



or $\lfloor v(x) \rfloor = c$ and $\{v(x)\} \neq 0$

Since $v \simeq_{nbd} v'$. $\lfloor v'(x) \rfloor = \lfloor v(x) \rfloor$

$\{v'(x)\} = 0$ iff $\{v(x)\} = 0$

if $\lfloor v(x) \rfloor \geq c+1$ then $\lfloor v'(x) \rfloor \geq c+1$

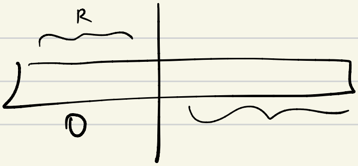
or if $\lfloor v(x) \rfloor = c$ $\wedge$ $\{v(x)\} \neq 0$ then $\lfloor v'(x) \rfloor = c$

$\{v'(x)\} \neq 0$

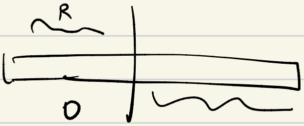
Lemma 3: Let $v \simeq_{nbd} v'$.

For every subset of clocks R: $[R]v \simeq_{nbd} [R]v'$ $v[R := 0]$

Proof:



$\simeq_{nbd}$



Follows since the $\simeq_{nbd}$ equivalence holds for clocks $X \setminus R$

and for clocks in R, both $v(x)$ and $v'(x)$ are 0.

$q \xrightarrow[R]{g,\ a} q'$

Proposition 4: Let $v \simeq_{nbd} v'$.

For every transition: $(q, v) \xrightarrow{\delta,\ a} (q_1, v_1)$

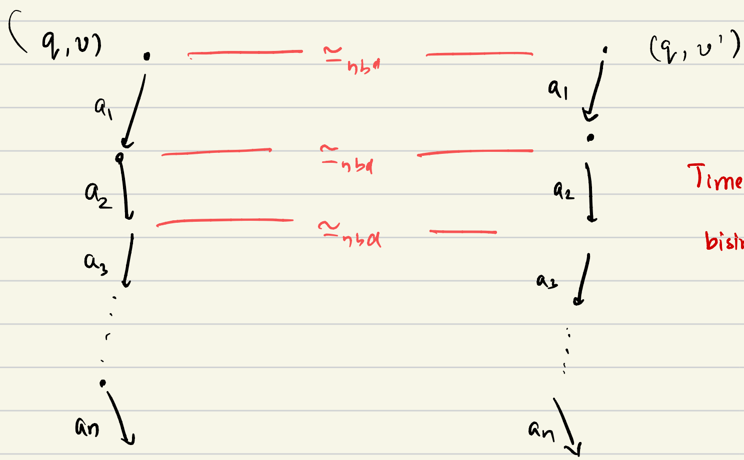
there exists a transition: $(q, v') \xrightarrow{\delta',\ a} (q_1, v_1')$

such that: $v_1 \simeq_{nbd} v_1'$

Proof:

Expanding $(q, v) \xrightarrow{\delta,\ a} (q_1, v_1)$:

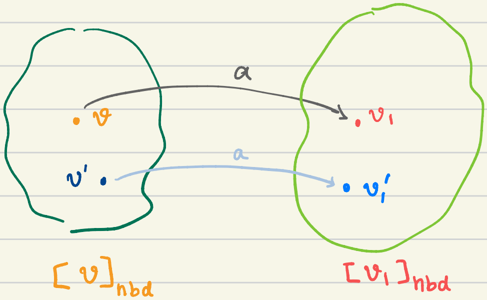
$$\begin{array}{ccccc}
(q, v) & \xrightarrow{\delta} & (q, v+\delta) & \xrightarrow[{[R]}]{g,\ a} & (q_1, v_1) \\
\simeq_{nbd} & & \simeq_{nbd} & & \simeq_{nbd} \\
(q, v') & \xrightarrow[\text{Lemma 1}]{\delta'} & (q, v'+\delta') & \xrightarrow[\text{Lemma 2, Lemma 3}]{} & (q_1, v_1)
\end{array}$$



$\simeq_{nbd}$

Time-abstract bisimulation

Illustration of Proposition 4:



$$\boxed{(q, [v]_{nbd}) \xrightarrow{a} (q_1, [v_1]_{nbd})}$$ to mean:

there exists δ s.t. $\Big\}$: $(q, v) \xrightarrow{\delta, a} (q_1, v_1)$

NBD (A): Neighbourhood automaton.

States: $(q, [v]_{nbd})$ $\quad Q \times$ Neighbourhoods

Initial state: $(q_0, [\vec{0}]_{nbd})$ $\quad q_0 \in Q_0$ (initial state of A)

Transitions:

$$(q, [v]_{nbd}) \xrightarrow{a} (q_1, [v_1]_{nbd})$$

if there exists a transition $(q, v) \xrightarrow{\delta, a} (q_1, v_1)$ in the semantics of A (S_A)

Final states: $(q, [v])$ where $q \in F$ (final)

Theorem: Nbd(A) accepts Untime($\mathcal{L}(A)$)

Proof: i) $\mathcal{L}(\mathrm{Nbd}(A)) \subseteq$ Untime$(\mathcal{L}(A))$

$$(q_0, [0]_{nbd}) \xrightarrow{a_1} (\quad) \xrightarrow{a_2} (\quad) \rightarrow \cdots \xrightarrow{a} (\quad , \quad)$$

ii) Untime$(\mathcal{L}(A)) \subseteq \mathcal{L}(\mathrm{Nbd}(A))$

$a_1 \; a_2 \; \ldots \; a_n$

there is a time word with an acc. run:

$$(q_0, v_0) \xrightarrow{\delta_1, a_1} (q_1, v_1) \xrightarrow{\delta_2, a_2} (q_2, v_2) \rightarrow \cdots \xrightarrow{a_n} (q_n, v_n)$$

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Next class.

2. Region equivalence $\longrightarrow$ aut. $Reg(\mathcal{A})$
   - accepts $Untime(\mathcal{L}(\mathcal{A}))$
   - finitely many states