TIMED AUTOMATA

LECTURE 23

Alternating Timed Automata:

- What we have seen so far?
  - Model is closed under union, intersection, complement
  - Emptiness is undecidable for general ATA
  - Consider 1-clock ATA
    - ↦ Expressive power incomparable to many clock NTA.

Today:
- Emptiness is decidable for 1-clock ATA (idea of proof)
- Complexity of the emptiness problem

Algorithm for the emptiness problem for 1-ATA:

Given a 1-clock ATA $\mathcal{A}$, is $\mathcal{L}(\mathcal{A})$ empty?

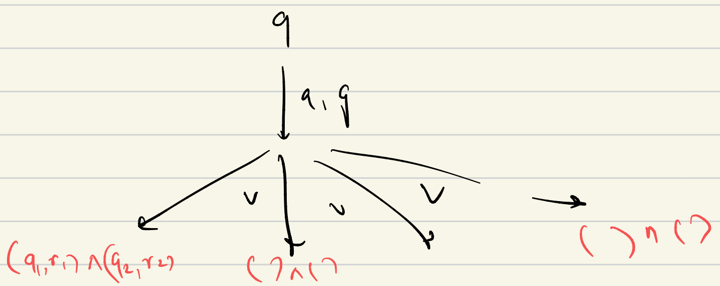
- Algorithm similar to Ouaknine-Worell algorithm for universality of 1-NTA
- Now we need to handle both universal and existential transitions.

Assumption:

- boolean combinations in the transitions are in

disjunctive normal form

$$(\cdot \wedge \cdot \wedge \cdot \cdot) \vee (\cdot \wedge \cdot \wedge \cdot \wedge \cdot\cdot) \vee \cdots \vee (\cdot \wedge \cdot \wedge \cdots \wedge)$$



Labelled transition system: $T(A)$

Configuration P: $\{ (q_1, v_1)\ (q_2, v_2), \ldots, (q_k, v_k) \}$

a set of states

(location of automaton, value of clock)

Transitions between configurations:

$$P \xrightarrow{t,a} P'$$

$P = \{(q_1, v_1)\ \ (q_2, v_2)\}$

t, a $\quad$ t, a

b_1 $\quad$ b_2

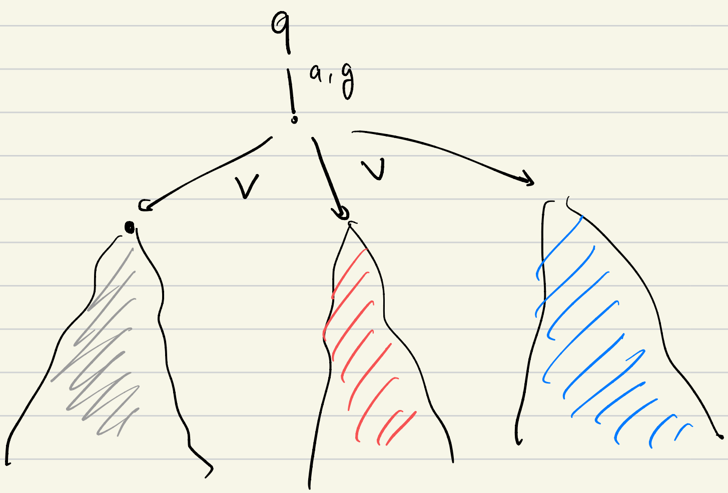
$b_1^1 \vee b_1^2 \vee b_1^3$ $\quad$ $b_2^1 \vee b_2^2$

For each $(q, v) \in P$

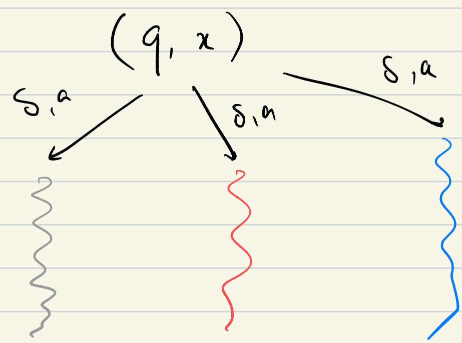
- let $v' = v + t$
- let $b = \delta(q, a, \sigma)$ for the uniquely determined σ satisfied by v'
- Choose one of the disjuncts of b: $(q_1, r_1) \wedge (q_2, r_2) \wedge \cdots \wedge (q_k, r_k)$
- $\text{Next}_{(q,v)} := \{ (q_i, v'[r_i := 0]) \mid i = 1, \ldots, k \}$

Then, $P' = \bigcup_{(q,v) \in P} \text{Next}_{(q,v)}$

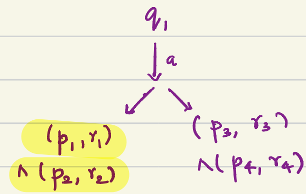
$\text{Next}^{b_1^1}_{(q_1,v_1)}$ $\quad$ $\text{Next}^{b_1^2}_{(q_1,v_1)}$ $\quad$ $\text{Next}^{b_1^3}_{q_1,v_1}$ $\quad \Bigg\| \quad$ $\text{Next}^{b_2^1}_{(q_2,v_2)}$ $\quad$ $\text{Next}^{b_2^2}_{(q_2,v_2)}$

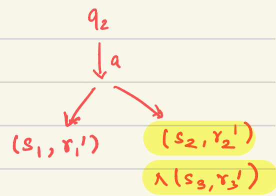


$x + \delta \models g$



$$\{(q_1, v_1), (q_2, v_2)\} \xrightarrow{a,\,t} \{(p_1, \cdot)(p_2, \cdot), (s_2, \cdot), (s_3, \cdot)\}.$$

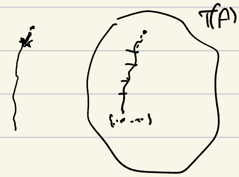




One possible transition in $T(\mathcal{A})$

Good node: All states are accepting

Theorem: $\mathcal{L}(\mathcal{A})$ is non-empty iff $T(\mathcal{A})$ has a path to a good node from the initial configuration



Rest of the algorithm similar to OW-05.

Lower bound

Complexity of emptiness of **purely universal** 1-clock ATA is **not** bounded by a **primitive recursive** function

Lower bound

Complexity of emptiness of **purely universal** 1-clock ATA is **not** bounded by a **primitive recursive** function

Emptiness of purely universal 1-ATA $\longrightarrow$ universality of 1-NTA

$A \longrightarrow A^c$ (1-NTA)

$\Rightarrow$ complexity of Ouaknine-Worrell algorithm for **universality** of 1-clock TA is **non-primitive recursive**

Primitive recursive functions

Functions $f : \cancel{\mathbb{N} \mapsto \mathbb{N}}$ $\underline{\underline{\mathbb{N}^k}} \longmapsto \underline{\underline{\mathbb{N}^\ell}}$ $k \geq 0$

Basic primitive recursive functions:

- **Zero function:** $Z() = 0$, Constant function: $C_n^k(x_1, \dots x_k) = n$
- **Successor function:** $Succ(n) = n + 1$
- **Projection function:** $P_i(x_1, \dots, x_n) = x_i$

$h \circ (g_1, g_2 \dots g_m)$

Operations:

$g_1(x_1 \dots x_k)$ $g_2(x_1 \dots x_k)$ $\dots g_m(x_1 \dots x_k)$

- **Composition** $h(y_1 \quad y_2 \quad y_m)$
- **Primitive recursion:** if f and g are p.r. of arity k and $k+2$, there is a p.r. h of arity $k+1$:

$$
\begin{aligned}
h(0, x_1, \dots, x_k) &= f(x_1, \dots, x_k) \\
h(n+1, x_1, \dots, x_k) &= g(h(n, x_1, \dots, x_k), n, x_1, \dots, x_k)
\end{aligned}
$$

$h(n, x_1, \dots x_k), n, (x_1 \dots x_k)$

Composition:

p.r.

$$g_1 : \mathbb{N}^k \longrightarrow \mathbb{N} \qquad g_1(x_1, \dots x_k) \longrightarrow y_1$$

$$\vdots$$

$$g_m : \mathbb{N}^k \longrightarrow \mathbb{N} \qquad g_m(x_1 \dots x_k) \longrightarrow y_m$$

$$h : \mathbb{N}^m \longrightarrow \mathbb{N}$$

$$h \circ (g_1, \dots g_m)\,[x_1 \dots x_k] \rightarrow h\,[g_1(x_1 \dots x_k), g_2(x_1 \dots x_k), \dots, g_m(x_1 \dots x_k)]$$

will be p.r. obtained by composition.

Addition:

h $\quad f(y) = y$

$$
\begin{aligned}
Add(0, y) &= y \\
h\ Add(n+1, y) &= Succ(Add(n, y))
\end{aligned}
$$

$Succ(P_1(Add(n, y), n, y))$

h: $Succ \circ P_1$

Addition:

$$
\begin{aligned}
Add(0, y) &= y \\
Add(n+1, y) &= Succ(Add(n, y))
\end{aligned}
$$

$Succ(P_1(Add(n,y), n, y))$

Multiplication:

$$
\begin{aligned}
Mult(0, y) &= Z() \\
Mult(n+1, y) &= Add(Mult(n, y), y)
\end{aligned}
$$

$$P_1(Mult(n,y), n, y) = Mult(n,y)$$

$$P_3(Mult(n,y), n, y) = y$$

$$Add \circ (P_1, P_3) : (Mult(n,y), n, y)$$

$$= Add(P_1(\ ), P_3(\ ))$$

$$= Add(Mult(n,y), y)$$

Addition:

$$\begin{aligned} Add(0, y) &= y \\ Add(n+1, y) &= Succ(Add(n, y)) \end{aligned}$$

Multiplication:

$$\begin{aligned} Mult(0, y) &= Z() \\ Mult(n+1, y) &= Add(Mult(n, y), y) \end{aligned}$$

Exponentiation 2^n:

$$\begin{aligned} Exp(0) &= Succ(Z()) \\ Exp(n+1) &= Mult(Exp(n), 2) \end{aligned}$$

$P_1[Exp(n), n]$

C_2^2

$Mult \circ (P_1, C_2^2)$

Addition:

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$HyperExp(1) = 2$

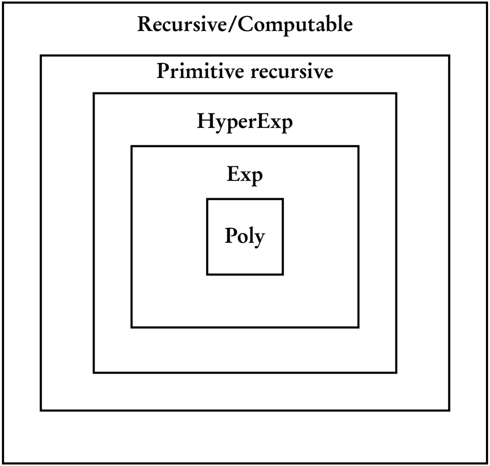
$HyperExp(2) = 2^2$

$(3) = 2^{2^2}$

$(4) = 2^{2^{2^2}}$

Hyper-exponentiation (tower of n two-s):

$$
\begin{aligned}
HyperExp(0) &= Succ(Z()) \\
HyperExp(n+1) &= Exp(HyperExp(n))
\end{aligned}
$$



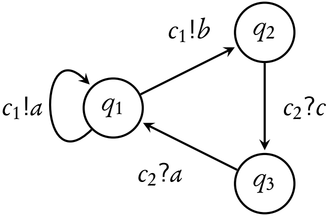
Recursive but not primitive rec.: Ackermann function, Sudan function

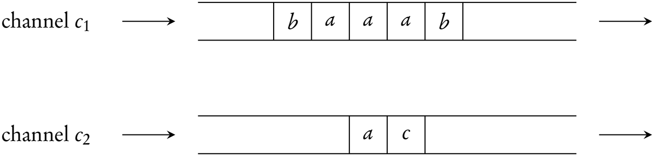
Coming next: a problem that has complexity non-primitive recursive

Channel systems

$(q, w) \xrightarrow{c!a} (q', aw)$

$(q, wa) \xrightarrow{c?a} (q', w)$





Finite state description of communication protocols

G. von Bochmann. 1978

On communicating finite-state machines

D. Brand and P. Zafiropulo. 1983

Theorem [BZ'83]

Reachability in channel systems is **undecidable**

Coming next: modifying the model for decidability

Lossy channel systems

Finkel'94, Abdulla and Jonsson'96

Messages stored in channel can be **lost** during transition

$(q, w) \xrightarrow{c!a} (q', w')$ where w' is a subword of aw

$(q, wa) \xrightarrow{c?a} (q', w'')$ where w'' is a subword of w

Lossy channel systems

Finkel'94, Abdulla and Jonsson'96

Messages stored in channel can be **lost** during transition

Theorem [Schnoebelen'2002]

Reachability for **lossy one-channel** systems is **non-primitive recursive**

Reachability problem for **lossy one-channel** systems can be reduced to emptiness problem for **purely universal 1-clock ATA**

1-clock ATA

- **closed** under boolean operations
- **decidable** emptiness problem
- expressivity **incomparable** to many clock TA
- **non-primitive recursive** complexity for emptiness

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- Other results: **Undecidability** of:
  - 1-clock ATA + ε-transitions
  - 1-clock ATA over infinite words