TIMED AUTOMATA

LECTURE 14

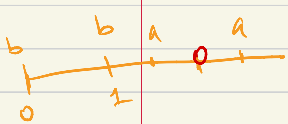
Solutions to Problem Sheet 4

1. $L_0 = \{(\sigma_1\sigma_2\ldots\sigma_k, \tau_1\ldots\tau_k)$ s.t.

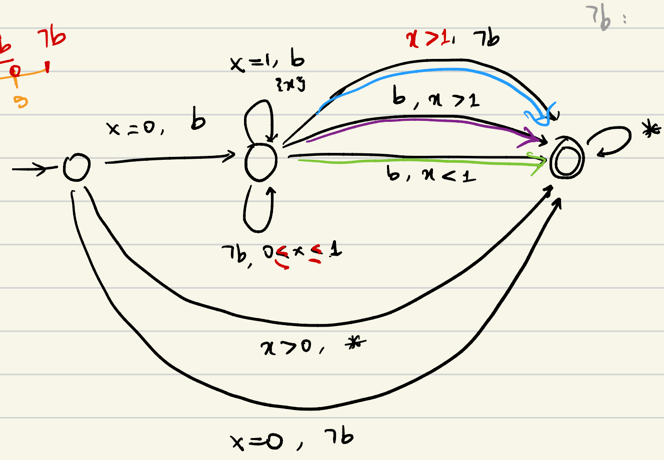
- a) either $\tau_1 > 0$
- b) or σ_1 is not a b-symbol
- c) or $\exists i \in \{1,\ldots,k\}$ s.t. τ_i is a b-symbol, but $\tau_i \notin \mathbb{N}$
- d) or $\exists j \in \mathbb{N}$ with $\tau_1 \leq j \leq \tau_k$ s.t. there is no b-symbol at j.

b: a b-symbol

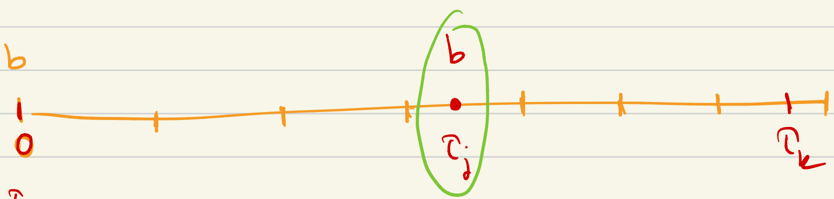
$\neg b$: not a b-symbol



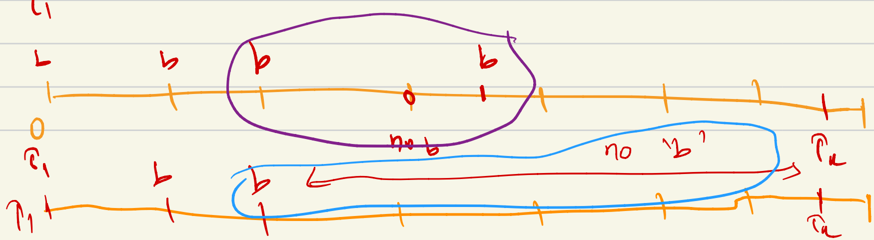
A_0:



(c)



(d)



2. Complement of $\mathcal{L}(A_0)$:

- $(\sigma_1 \sigma_2 \ldots \sigma_k, \tau_1 \tau_2 \ldots \tau_k)$ s.t.

a) $\tau_1 = 0$ and

b) there is a b-symbol at every integer $\leq \tau_k$

c) there is no b-symbol at non-integer timestamps $\leq \tau_k$.

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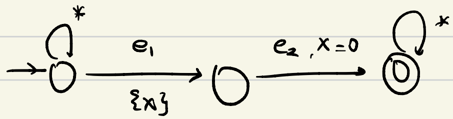
3. A_0':



for all combinations of $e_1, e_2 \in \Sigma_{enc}$

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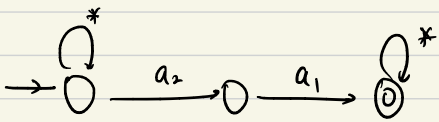
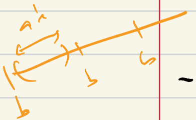
4. A_1:



5. Complement of $\mathcal{L}(A_0) \cup \mathcal{L}(A_0') \cup \mathcal{L}(A_1)$



Timed words $(\sigma_1 \sigma_2 \ldots, \sigma_k, \tau_1 \tau_2 \ldots \tau_k)$ s.t.

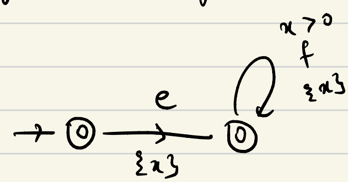
- there is a b-symbol at every integer $\leq \tau_k$
- in every interval $(i, i+1)$ where $i \in \mathbb{N}$ and $i \leq \tau_k$, the string is of the form $a_1^* a_2^*$
- no two letters appear at the same time-stamp.

6. Yes.

- Complement of $\mathcal{L}(A_0) \cup \mathcal{L}(A_0') \cup \mathcal{L}(A_1)$ is $[\mathcal{L}(A_0)]^c \cap [\mathcal{L}(A_0')]^c \cap [\mathcal{L}(A_1)]^c$

- Since A_0 is deterministic and complete, the complement is obtained by interchanging acc & reject states.

- Complement of A_0':



$\forall e, f \in \Sigma_{enc}$

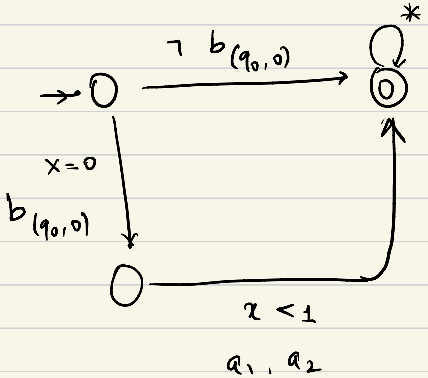
- Since A_1 is a "regular language" complement can be obtained by considering its DFA.

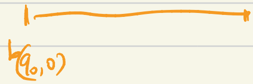
7. Initial configuration of M: q_0 (initial state)

tape head $= 0$

$C = 0$

$D = 0$





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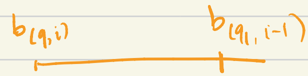
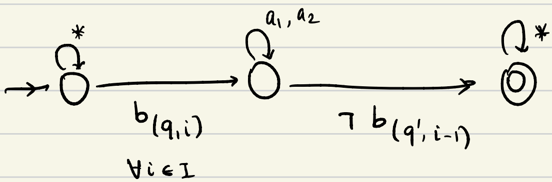
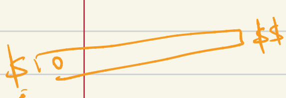
8. $t : (q, 0, D{+}{+}, L, q_1)$

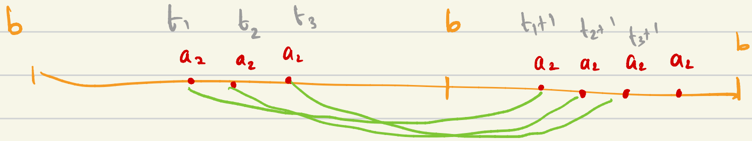


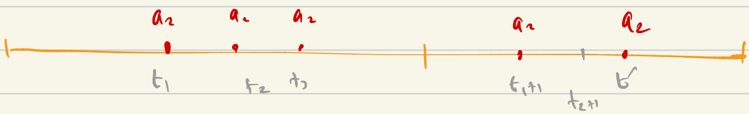
Let $I = \{ i \in \{1, \ldots, k\} \mid w_i = 0 \}$



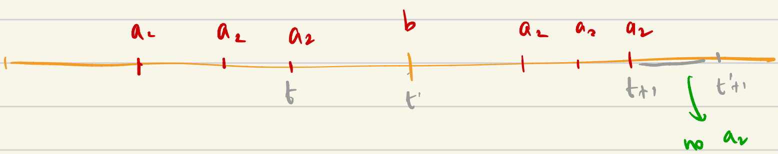
Violation of state and tapehead position.



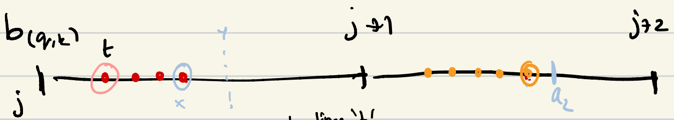

b
t1 t2 t3
b
t1+1 t2+1 t3+1
b
a2 a2 a2 a2 a2 a2 a2


a2 a2 a2 a2 a2
t1 t2 t3 t1+1 t2+1 t'

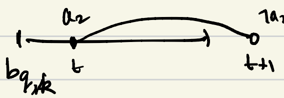

a2
a2 a2 a2 a2 a2 a2
t1 t2 t3 t1+1 t2+1 t2+1

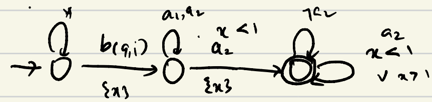

a2 a2 a2 b a2 a2 a2
t t' t+1 t'+1
no a2

Violation of counter update.

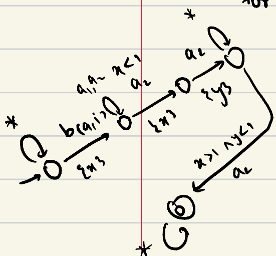


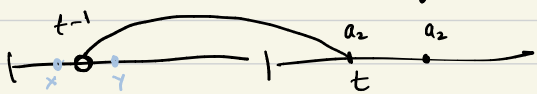
-1) there is some a_2 after $b_{(q,k)}$ at time 't' such that there is no 'a_2' at $t+1$





-2) there is an 'a_2' in $t \in [j+1, j+2)$ which is not the last a_2 for which there is no corresponding a_2 at $t-1$.

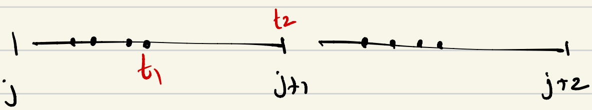


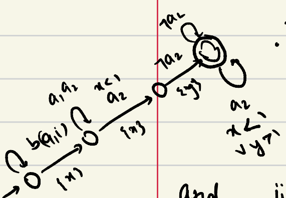


- Guess 2 positions around '$t-1$' and reset 2 clocks $\{x\}, \{y\}$

- Read an a_2 at $x > 1 \wedge y < 1$ $\boxed{2 \text{ clocks}}$

3)





i) $\forall a_2$ in $t \in (j, j+1)$ there exists an a_2 at $t+1$

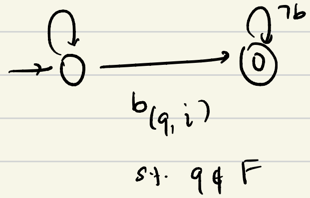
and ii) Suppose t_1 is the timestamp of the last a_2 in $(j, j+1)$

and t_2 is the time stamp of the next letter.

then there is no 'a_2' in (t_1+1, t_2+1)

- can also be done with 2 clocks.

9.



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10. Complement of $\mathcal{L}(A_0) \cup \mathcal{L}(A_0') \cup \mathcal{L}(A_1) \cup \mathcal{L}(A_{init}) \cup$

$$\bigcup_{t \in \delta} \mathcal{L}(A_t) \cup \mathcal{L}(A_{cc})$$

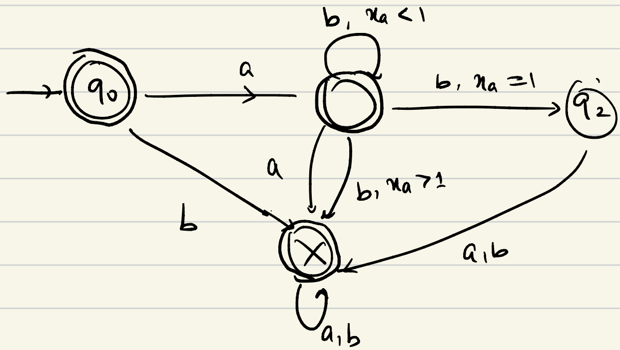
is the set of all words $(\sigma_1 \sigma_2 \ldots \sigma_k, \tau_1 \ldots \tau_k)$ where

- no two symbols appear at the same time-stamp
- there is a 'b' at every natural number $\leq \tau_k$
- between two b's the string is of the form $a_1^* \, a_2^*$
- part of the word in $[0,1)$ encodes initial config. of M
- successive unit intervals encode transitions of M
- accepting config is encoded in the end

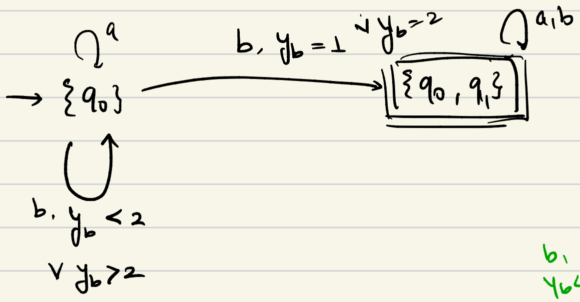
- This equals L_{undec}

Solutions to Problem Sheet 5

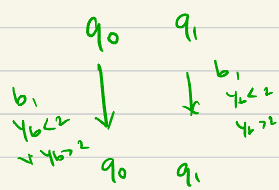
1. Determinize, complete the ECA and swap acc\rej states:



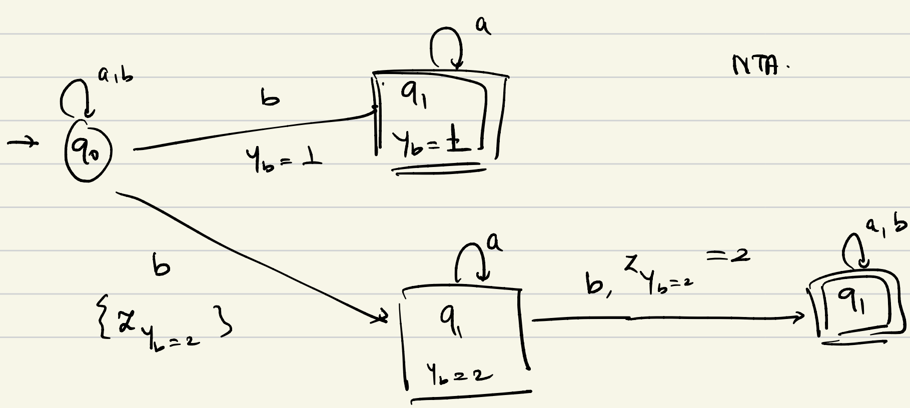
2.



DECA.



NTA.



3.

a) No:

$\Sigma = \{a, b\}$ $\quad \Gamma = \{c\}$

$h : a \mapsto c$

$\quad\; b \mapsto c$

$L = \{ abb \mid$ distance bet. a and b is 1 $\}$

L has an ERA

$h(L) = \{ ccc \mid$ distance bet. first & last 'c' is 1 $\}$

$h(L)$ has no ECA.

b) $L : \{ a^n b^n \mid n \geq 0 \}$ $\quad L$ is not ECA-language

$h : a \mapsto c$

$\quad\; b \mapsto c$

$h(L) = \{ c^{2n} \mid n \geq 0 \}$ $\quad$ a regular language,

– can also be considered as ECA.

Exercise: Regular languages are closed under inverse homomorphisms. Why does above example not contradict this theorem?

Solutions to Problem Sheet 6:

1) $\to \bigcirc \xrightarrow{x \geq 2} \bigcirc \xrightarrow{x \leq 1} \circledcirc$

2) No. Consider a run to an accepting state which elapses no time:

$$(q_0, v_0) \xrightarrow{\delta_0, t_0} (q_1, v_1) \xrightarrow{\delta_1, t_1} \cdots \longrightarrow (q_n, v_n)$$

$$\delta_i = 0$$

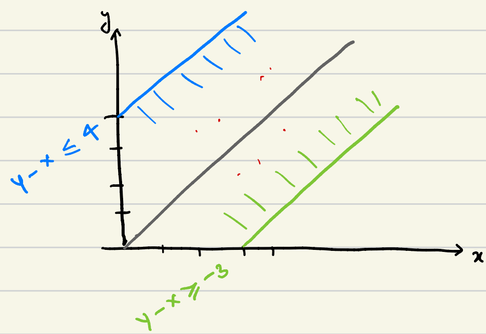
Every clock has value 0 all along the run.

Since we have only upper bound constraints $x < c$, $x \leq c$, the guards are satisfied and the run is indeed valid.

3) No. Let M be the max constant occurring in guards.

Elapse M+1 time units at each transition.

4)



No. We cannot get the above zone during the forward analysis zone algorithm.

1) $\bullet \sim\sim\sim \bullet Z$ $\quad (x = y)$ if neither x nor y is reset or if both are reset at the same time.

$\boxed{\begin{array}{c|c} x & y \\ \geq 0 & 0 \end{array}}$

2) $\bullet \sim\sim \{x\} \sim\sim \{y\} \sim\sim \bullet Z \leq (y \leq x)$

In the forward analysis algorithm, the zones that are reached remember the order of resets along the path leading to the zone.

So the zone $-3 \leq y - x \leq 4$ cannot be reached because it contains some valuations where $x < y$ and some others where $y < x$.

More formally: A zone is totally ordered if for every pair of clocks x, y,

either $v(x) \leq v(y)$ for all $v \in Z$

or $v(y) \leq v(x)$ for all $v \in Z$.

Following operations preserve total order:

1) Time elapse: Z is totally ordered $\Rightarrow$ $\overrightarrow{Z}$ is totally ordered
2) guard intersection: Z is totally ordered $\Rightarrow$ $Z \cap g$ is totally ordered.
3) Reset : Z is totally ordered $\Rightarrow$ $[R]Z$ is totally ordered.

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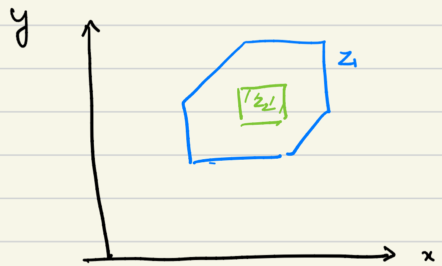
5. No.



$Z_1 - Z_2$ is a disconnected set.

Whereas a zone is always a convex set.

Zone: "conjunction" of $x - y \sim c$
$x \sim c$

Why is a zone convex?

$\bar{v}, \bar{u} \in Z$

$\lambda \bar{v} + (1-\lambda)\bar{u}$

Consider for instance:

$x - y \leq 5$

(x_1, y_1) (x_2, y_2) s.t. $x_1 - y_1 \leq 5$

$x_2 - y_2 \leq 5$

$\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2$:

$\lambda(x_1 - y_1) + (1-\lambda)(x_2 - y_2) \leq \lambda 5 + (1-\lambda)5$

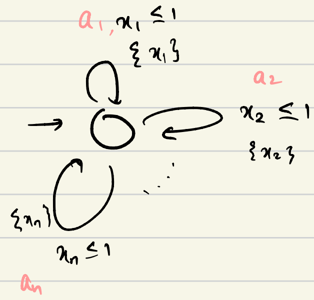
≤ 5

Similar argument holds for $x - y \geq c$ constraints.

$x \leq c$

$x \geq c$

6.

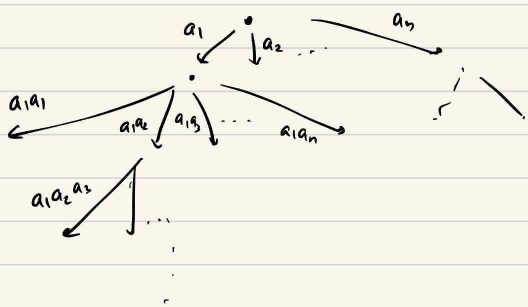


Let σ be a sequence over $\{a_1, a_2, \ldots, a_n\}$

We denote by Z_σ the zone reached by the transition sequence σ.

For eg: $Z_{a_1 a_2}$ is zone after $\xrightarrow[\{x_1\}]{x_1 \leq 1} \xrightarrow[\{x_2\}]{x_2 \leq 1}$.

The zone graph computation of the above automaton looks like this:



Let us consider sequences σ with no repetitions, for eg. ignore $a_1 a_1$
$a_1 a_2 a_1$, etc.

Consider two sequences σ_1 and σ_2, possibly of diff. lengths.

<u>Case i)</u> Suppose there are two events a_i, a_j s.t.

$$a_i \in \sigma_1 \quad \text{but} \quad a_i \notin \sigma_2$$
$$a_j \notin \sigma_1 \quad\quad\quad a_j \in \sigma_2 .$$

In Z_{σ_1} we will have $x_i \leq x_j$, and in Z_{σ_2} $x_j \leq x_i$

Consider a run which elapses 1 time unit at the beginning and then takes σ_1 with no time elapse.

~ This run reaches a valuation v_1 with $v_1(x_i) = 0$
$v_1(x_j) = 1$

We have $v_1 \in Z_{\sigma_1}$

But no valuation region equivalent to v_1 belongs to Z_{σ_2} as $x_j \leq x_i$ in Z_{σ_2}.

$\therefore$ $Z_{\sigma_1} \not\preccurlyeq_M Z_{\sigma_2}$

- By a similar argument we can show that $Z_{\sigma_2} \not\preccurlyeq_M Z_{\sigma_1}$

<u>Case ii</u> suppose there are two events a_i, a_j s.t.

- a_i comes after a_j in σ_1 $\quad$ a_j comes after a_i in Z_{σ_2}

Similarly we will have $x_i \leq x_j$ in Z_{σ_1} and $x_j \leq x_i$ in Z_{σ_2}

$\sigma_1 : \quad \xrightarrow{\quad a_j \quad} \xrightarrow{a_i}$ ———

Consider a run that elapses no time upto a_i, then elapses one time unit before taking a_i, and then again no time at all after this.

In the end we will have a valuation v_1 with:

$v_1(x_j) = 1$
$v_1(x_i) = 0$

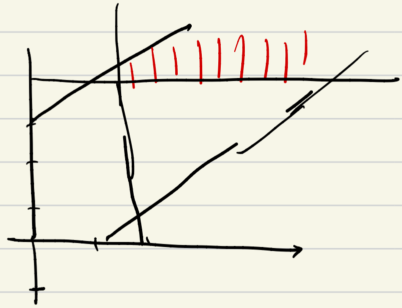
Similar to case (i), $v_1 \in Z_{\sigma_1}$, but no region equivalent valuation belongs to Z_{σ_2}

$$Z_{\sigma_1} \not\approx_M Z_{\sigma_2}.$$

Analogously, $Z_{\sigma_2} \not\approx_M Z_{\sigma_1}$.

7.

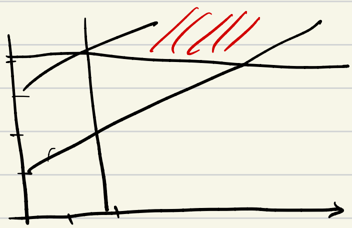
$$y - x \leq 3$$
$$x - y \leq -1$$
$$x \geq 2$$
$$y \geq 4$$



→ Not totally ordered

So not reachable.

$$y - x \leq 3$$
$$x - y \leq 1$$
$$x \geq 2$$
$$y \geq 4$$



Totally ordered.

$$\bullet \xrightarrow[\{x\}]{1 \leq y \leq 3} \circ \xrightarrow{x \geq 2 \ \wedge y \geq 4} \bullet$$

leads to this zone.

8. No.

In the zone enumeration, each zone is of the form $x \geq c$ or $x > c$.

Therefore from (q, z) at some point a node (q_1, z_1) is reached s.t. $z_1 \subseteq z$.