The goal of this problem sheet is to better understand the details in the proof of undecidability of universality for timed automata as presented in Lecture 6. We will write $\mathbb{N}$ for the set of natural numbers.

Counter machine M**.** We are given a deterministic 2-counter machine $M = (Q, \{0,1\}, \{C, D\}, \delta, F, \{\$, \$\$\})$ where Q is a finite set of states, $\{0,1\}$ is the input alphabet, \$ is the left end-marker and \$\$ is the right end-marker for the input word, $\{C, D\}$ are counters, $F \subseteq Q$ is a set of accepting states and $\delta \subseteq Q \times \{0, 1, \$, \$\$\} \times \{C = 0?, D = 0?, \varepsilon\} \times \{C++, C--, \varepsilon\} \times \{D++, D--, \varepsilon\} \times \{L, R\} \times Q$ is a transition relation. In each transition, the counters can be checked for 0, and can be incremented or decremented or left unchanged, and the tape head can move either left or right. We will assume that when \$ is read, the tape-head can only move right and when \$\$ is read, the tape can only move to the left.

Word w. We are also given a word $w \in \{0,1\}^*$. Length of the w is denoted as $|w|$.

Encoding as a timed language. For the encoding of the accepting computations of M on w, we make use of an alphabet Σ_{enc} which contains a $b_{(q,w_i)}$ for each $q \in Q$ and $i \in \{0, 1, \ldots, |w| + 1\}$, and two letters $\{a_1, a_2\}$. We will call the letters of the form $b_{(q,w_i)}$ as b-symbols, and a_1 and a_2 as a-symbols.

We will make use of the language L_{undec} and its complement $\overline{L}_{undec}$ discussed in Lecture 6 for the questions below.

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1. Construct a timed automaton $\mathcal{A}_0$ over Σ_{enc} which accepts all timed words $(\sigma_1\sigma_2 \ldots \sigma_k, \tau_1\tau_2 \ldots \tau_k)$ such that one of the following conditions is true:
   (a) either $\tau_1 > 0$
   (b) or σ_1 is not a b-symbol
   (c) or there exists an $i \in \{1, \ldots, k\}$ such that τ_i is a b-symbol, but $\tau_i \notin \mathbb{N}$,
   (d) or there exists some $j \in \mathbb{N}$ with $\tau_1 \leq j \leq \tau_k$ such that there is no b-symbol at j.
2. What is the complement of $\mathcal{L}(\mathcal{A}_0)$?
3. Construct a timed automaton $\mathcal{A}_0'$ over Σ_{enc} that accepts all timed words in which there are two letters appearing at the same time-stamp.
4. Construct a timed automaton $\mathcal{A}_1$ over Σ_{enc} that accepts all words that contain a_2a_1 as substring.
5. What is the complement of $\mathcal{L}(\mathcal{A}_0) \cup \mathcal{L}(\mathcal{A}_0') \cup \mathcal{L}(\mathcal{A}_1)$?
6. Let L_{pre} be the complement of $\mathcal{L}(\mathcal{A}_0) \cup \mathcal{L}(\mathcal{A}_0') \cup \mathcal{L}(\mathcal{A}_1)$. Can you construct a timed automaton for L_{pre}?
7. Construct a timed automaton $\mathcal{A}_{init}$ which accepts all words in $\overline{L}_{undec} \cap L_{pre}$ in which the interval $[0, 1)$ does not encode the initial configuration of M.
8. Let $t : (q, 0, D++, L, q_1)$ be a transition of M. Construct a timed automaton $\mathcal{A}_t$ which accepts all words in $\overline{L}_{undec} \cap L_{pre}$ where t is violated.
9. Construct a timed automaton $\mathcal{A}_{acc}$ which accepts all words where the final b-symbol corresponds to a non-accepting state of M.
10. Show that $\overline{L}_{undec}$ is equal to $\mathcal{L}(\mathcal{A}_0) \cup \mathcal{L}(\mathcal{A}_0') \cup \mathcal{L}(\mathcal{A}_1) \cup \mathcal{L}(\mathcal{A}_{init}) \cup \bigcup_{t \in \delta} \mathcal{L}(\mathcal{A}_t) \cup \mathcal{L}(\mathcal{A}_{acc})$.