TIMED AUTOMATA

LECTURE 4

GOALS OF TODAY'S LECTURE:

1. Neighbourhood equivalence $\longrightarrow$ aut. $NBD(\mathcal{A})$
   - accepts $\mathrm{Untime}(\mathcal{L}(\mathcal{A}))$
   - infinitely many states

2. Region equivalence $\longrightarrow$ aut. $Reg(\mathcal{A})$
   - accepts $\mathrm{Untime}(\mathcal{L}(\mathcal{A}))$
   - finitely many states

Definition 1 (Neighbourhood equivalence) Two valuations v and v' are said to be neighbourhood equivalent, written as $v \simeq_{\mathsf{nbd}} v'$ if:

1. $\lfloor v(x) \rfloor = \lfloor v'(x) \rfloor$ for all clocks x,
2. $\{v(x)\} = 0$ iff $\{v'(x)\} = 0$ for all clocks x
3. for every pair of clocks x, y:
   (a) $\{v(x)\} < \{v(y)\} \Leftrightarrow \{v'(x)\} < \{v'(y)\}$
   (b) $\{v(x)\} = \{v(y)\} \Leftrightarrow \{v'(x)\} \underset{=}{\cancel{\neq}} \{v'(y)\}$

Each equivalence class of $\simeq_{\mathsf{nbd}}$ will be called a *neighbourhood*.

---

1. Group the following valuations over five clocks into neighbourhoods.

v_1, v_4 $\quad v_3 \quad v_2, v_6 \quad v_5$

(i) (ii) (iii) (iv) (v)

$$
\begin{aligned}
v_1 &:= (7.4,\ 2.1,\ 8.7,\ 5.4,\ 7.0) \rightarrow \begin{bmatrix} 7 & 2 & 8 & 5 & 7 \end{bmatrix} \\
v_2 &:= (3.4,\ 2.0,\ 8.5,\ 10.0,\ 7.1) \\
v_3 &:= (7.3,\ 2.2,\ 8.8,\ 5.2,\ 7.0) \\
v_4 &:= (7.5,\ 2.1,\ 8.9,\ 5.5,\ 7.0) \\
v_5 &:= (3.2,\ 2.0,\ 8.8, 10.0,\ 7.5) \\
v_6 &:= (3.3,\ 2.0,\ 8.4,\ 10.0,\ 7.2)
\end{aligned}
$$

(v) (ii) (i) (iv) (iii)

3 2 8 10 7

(iv) (ii) (v) (i) (iii)

(ii) (iv) (i) (v) (iii)

2. Below are combinations of v, δ and v' with $v \simeq_{\mathsf{nbd}} v'$. For each of them, find a $\delta' \geq 0$ such that $v + \delta \simeq_{\mathsf{nbd}} v' + \delta'$.

$$v = (1.1,\ 2.0,\ 3.0,\ 4.0,\ 5.9) \qquad \delta = 3.4 \qquad v' = (1.5,\ 2.0,\ 3.0,\ 4.0,\ 5.6)$$
$$v = (10.2,\ 4.8,\ 19.1,\ 2.0,\ 8.5) \qquad \delta = 5.4 \qquad v' = (10.1,\ 4.9,\ 19.05,\ 2.0,\ 8.7)$$

Recall that $v + \delta$ is the valuation obtained by adding δ to each coordinate of v.

$v = (1.1, 2.0, 3.0, 4.0, 5.9)$ $\quad \delta = 3.4 \quad v + \delta =$

$3 + 0.4$

1 2 3 4 5 $\xrightarrow{+3}$ 4 5 6 7 8 +1

0.1 ⇢ 0.5 $\quad$ 0.9 ⇢ $\quad$ +0.4 $\quad$ 0.3

4.5 5.0 6.0 7.0 9.3

$v' = (1.5, 2.0, 3.0, 4.0, 5.6)$ $\quad \delta' = 3.41 \qquad 3.4 < \delta < 3.5$

0.5 0.6

$v = (10.2, 4.8, 19.1, 2.0, 8.5)$ $\quad \delta = 5.4$

$v' = (10.1, 4.9, 19.05, 2.0, 8.7)$ $\quad \delta' = 5.2$

0.0 0.2 0.5 0.8

0.1 0.7 0.9

$0.1 < \delta < 0.3$

THREE OBSERVATIONS ABOUT

Neighbourhood equivalence

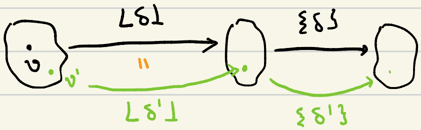
$\simeq_{nbd}$

Lemma 1: Let $v \simeq_{nbd} v'$.

For every $\delta \geq 0$, there exists $\delta' \geq 0$ s.t. $v+\delta \simeq_{nbd} v'+\delta'$

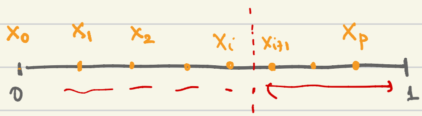
Proof: Constructing δ'. Firstly $\delta' = \lfloor \delta' \rfloor + \{\delta'\}$ $(3.4 = 3 + 0.4)$

Choosing $\lfloor \delta' \rfloor$: $\lfloor \delta' \rfloor = \lfloor \delta \rfloor$



Choosing $\{\delta'\}$: $v + \lfloor \delta \rfloor \simeq_{nbd} v' + \lfloor \delta' \rfloor$ (since fractional parts do not change by adding an integer)

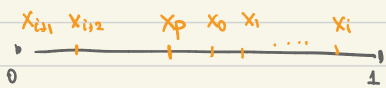
Let us look at the order of fractional parts in $v + \lfloor \delta \rfloor$ (or just v)



$1 - \{x_{i+1}\}^{v} \leq \{\delta\} \leq 1 - \{x_i\}^{v}$

$x_i \subseteq X$, $\bigcup x_i = X$

By adding $\{\delta\}$



We know that fractional parts in $v' + \lfloor \delta' \rfloor$ (or v') have the same order as above.

1) If $\{\delta\} = 1 - \{v_{x_{i+1}}\}$, then $\{\delta'\} := 1 - \{v'_{x_{i+1}}\}$

2) If $1 - \{v_{x_{i+1}}\} < \{\delta\} < 1 - \{v_{x_i}\}$, then $\{\delta'\} :=$ some val. between $1 - \{v'_{x_{i+1}}\}$ and $1 - \{v'_{x_i}\}$

3) If $\{\delta\} = 1 - \{v_{x_i}\}$ then $\{\delta'\} := 1 - \{v'_{x_i}\}$

Lemma 2: Let $v \simeq_{nbd} v'$.

For every guard φ : v satisfies φ iff v' satisfies φ.

(constants are natural nos in guards)

$$x < 5 \ \wedge\ y = 2 \ \wedge\ z \geq 4 \ \wedge\ w \geq 3$$

v: 4.8

3.2

3.$\cancel{0}$ 3

Proof: Follows since: $\left.\begin{array}{l} \lfloor v(x) \rfloor = \lfloor v'(x) \rfloor \\ \\ \text{and } \{v(x)\} = 0 \text{ iff } \{v'(x)\} = 0 \end{array}\right\} \forall$ clocks x.

Lemma 3: Let $v \simeq_{nbd} v'$.

For every subset of clocks R : $[R]v \simeq_{nbd} [R]v'$

Proof: Can be checked that each condition of $\simeq_{nbd}$ holds between $[R]v$ and $[R]v'$

$q \xrightarrow[R]{g,\, a} q'$

Proposition 4: Let $v \simeq_{nbd} v'$.

For every transition: $(q, v) \xrightarrow{\delta,\, a} (q_1, v_1)$

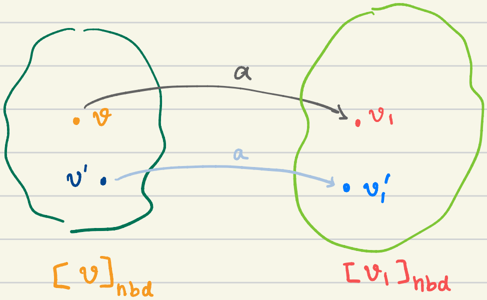
there exists a transition: $(q, v') \xrightarrow{\delta',\, a} (q_1, v_1')$

such that: $v_1 \simeq_{nbd} v_1'$

Proof: Expanding $(q, v) \xrightarrow{\delta,\, a} (q_1, v_1)$:

$$
\begin{array}{ccccc}
(q, v) & \xrightarrow{\delta} & (q, v+\delta) & \xrightarrow[{[R]}]{g,\, a} & (q_1, v_1) \\
\simeq_{nbd} & & \simeq_{nbd} & & \simeq_{nbd} \\
(q, v') & \xrightarrow[\text{Lemma 1}]{\delta'} & (q, v'+\delta') & \xrightarrow[\text{Lemma 2, Lemma 3}]{} & (q_1, v_1)
\end{array}
$$

Illustration of Proposition 4:



$\boxed{(q, [v]_{nbd}) \xrightarrow{a} (q_1, [v_1]_{nbd})}$ to mean:

there exists δ s.t. $\Big\}$: $(q, v) \xrightarrow{\delta, a} (q_1, v_1)$

NBD (A): Neighbourhood automaton.

States: $(q, [v]_{nbd})$ $Q \times$ Neighbourhoods

Initial state: $(q_0, [\vec{0}]_{nbd})$ $q_0 \in Q_0$ (initial state of $\mathcal{A}$)

Transitions:

$$(q, [v]_{nbd}) \xrightarrow{a} (q_1, [v_1]_{nbd})$$

if there exists a transition $(q, v) \xrightarrow{\delta, a} (q_1, v_1)$ in the semantics of $\mathcal{A}$ $(S_{\mathcal{A}})$

Final states: $(q, [v])$ where $q \in F$ (final)

Theorem: Nbd($\mathcal{A}$) accepts Untime($\mathcal{L}(\mathcal{A})$)

Proof: i) $\mathcal{L}(\text{Nbd}(\mathcal{A})) \subseteq \text{Untime}(\mathcal{L}(\mathcal{A}))$

$$(q_0, [0]_{nbd}) \xrightarrow{a_1} (\quad) \xrightarrow{a_2} (\quad) \to \cdots \xrightarrow{a} (\quad , \quad)$$

ii) $\text{Untime}(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{L}(\text{Nbd}(\mathcal{A}))$

$a_1 \; a_2 \ldots a_n$

there is a time word with an acc. run:

$$(q_0, v_0) \xrightarrow{\delta_1, a_1} (q_1, v_1) \xrightarrow{\delta_2, a_2} (q_2, v_2) \to \cdots \xrightarrow{a_n} (q_n, v_n)$$

GOALS OF TODAY'S LECTURE:

1. Neighbourhood equivalence $\longrightarrow$ aut. NBD($\mathcal{A}$)
   - accepts Untime($\mathcal{L}(\mathcal{A})$)
   - infinitely many states

2. Region equivalence $\longrightarrow$ aut. Reg($\mathcal{A}$)
   - accepts Untime($\mathcal{L}(\mathcal{A})$)
   - finitely many states

REGION EQUIVALENCE:

Maximum constant M: the maximum constant appearing in the guards of $\mathcal{A}$.

Intuition: When a clock crosses max constant M, its exact value (neighbourhood) is irrelevant.

$$\text{Bounded}(v) = \{x \mid v(x) \leq M\}$$

$$\text{Bounded}(v') = \{x \mid v'(x) \leq M\}$$

$$v \simeq_M v' \quad \text{if}$$

- 1. $\text{Bounded}(v) = \text{Bounded}(v')$
- 2. $v\big|_{\text{Bounded}(v)} \simeq_{nbd} v'\big|_{\text{Bounded}(v')}$

M=10 :

$$(13.1, \ 5.5, \ 7.8, \ 2.0, \ 100, \ 10.1)$$

$$(10.05, \ 5.6, \ 7.9, \ 2.0, \ 20, \ 1000)$$

Bounded clocks!

THREE OBSERVATIONS ABOUT

Region equivalence

$\simeq_M$

Lemma 5: Let $v \simeq_M v'$.

For every $\delta \geq 0$, there exists $\delta' \geq 0$ s.t. $v + \delta \simeq_M v' + \delta'$

Lemma 6: Let $v \simeq_M v'$.

For every guard φ with constant $\leq M$: v satisfies φ iff v' satisfies φ.

Lemma 7: Let $v \simeq_M v'$.

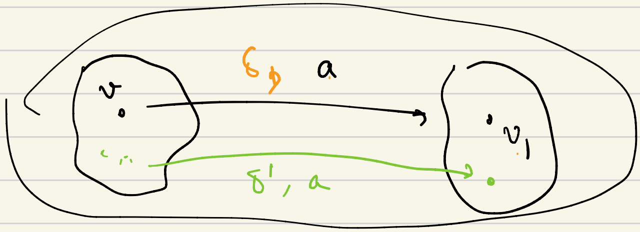
For every subset of clocks R: $[R]v \simeq_M [R]v'$

Proposition 8: Let $v \simeq_M v'$.

For every transition: $(q, v) \xrightarrow{\delta, a} (q_1, v_1)$

there exists a transition: $(q, v') \xrightarrow{\delta', a} (q_1, v_1')$

such that: $v_1 \simeq_M v_1'$



REGION AUTOMATON REG(A):

States: (q, r)

$\hookrightarrow$ region (equivalence class of $\simeq_M$)

Transitions:

$$(q, r) \xrightarrow{a} (q_1, r_1)$$

if there exists a valuation $v \in r$ s.t.

there exists a transition: $(q, v) \xrightarrow{\delta, a} (q_1, v_1)$ in the semantics S_A

s.t. $v_1 \in r_1$

Initial and final states are as before.

Theorem: Reg(A) is a finite automaton accepting Untime($\mathcal{L}(A)$).

1) x • • • • • •

0

(0,1)

1

⋮

(M−1, M)

M

(M, ∞)

2) Choose a subset with fractional part 0. $(2^{|X|})$

3) Ordering of fractional parts among bounded clock

$$(M+1)^{|X|} \cdot 2^{|X|} \cdot |X|!$$