HISTORY DETERMINISM

Lecture 8: Characterizing History-Determinism in coBüchi automata

Reference: Chapter 4 of Keya Prakash's thesis.

The goal of today's lecture is to prove the following theorem.

Theorem: For every coBüchi automaton $\mathcal{A}$, Eve wins the Joker game on $\mathcal{A}$ iff $\mathcal{A}$ is history-deterministic.

We have already seen this direction previously:

$\mathcal{A}$ is HD $\Rightarrow$ Eve wins 2-token game $\Rightarrow$ Eve wins Joker game.

To show: Eve wins Joker game $\Rightarrow$ $\mathcal{A}$ is HD

Recall two previous results:

- 1. Eve wins Joker game on $\mathcal{A}$ $\Rightarrow$ There is a simulation-equivalent subautomaton $\mathcal{B}$ where Eve wins 1-token game from everywhere.

- 2. If $\mathcal{A}$, $\mathcal{B}$ are simulation equivalent then: $\mathcal{A}$ is HD iff $\mathcal{B}$ is HD.

Therefore it is sufficient to show the following theorem.

Theorem: Let $\mathcal{A}$ be a coBüchi automaton on which Eve wins the 1-token game from everywhere.

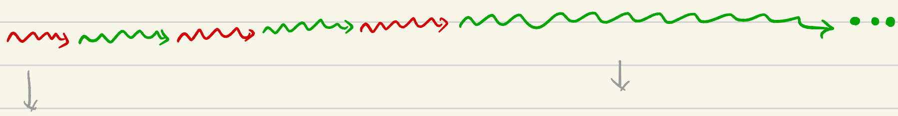
Then, $\mathcal{A}$ is HD.

CoBüchi condition: [1,2] parity condition

recall: parity condition asks min parity seen infinitely often is even.

Therefore, with [1,2], this means:

- priority 1 is seen only finitely often.
- Eventually, all transitions are 2, in an accepting run.



red transitions represent 1.

green transitions represent '2'

To show: $\mathcal{A}$ is a [1,2] - automaton s.t. Eve wins 1- token game from everywhere.

To show that $\mathcal{A}$ is HD.

ie, Eve has a strategy to win the HD game.

Step 1: Modify the automaton in a manner that accepting runs are preserved.

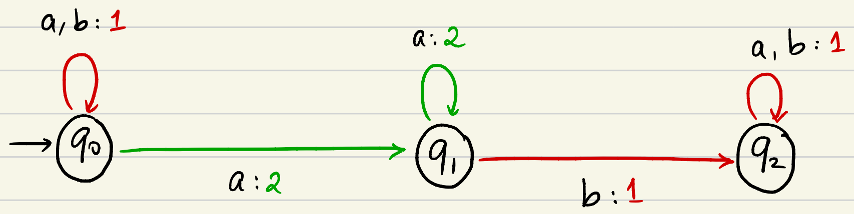
$$\mathcal{A} \longrightarrow G_{>1}$$

$\downarrow$

- Graph whose vertices are states of $\mathcal{A}$.
- All 2-transitions of $\mathcal{A}$ are edges in $G_{>1}$

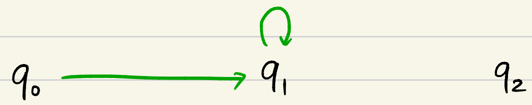
Example:

$\mathcal{A}_3$:



Accepts finitely many a's.

$G^{\mathcal{A}_3}_{>1}$:



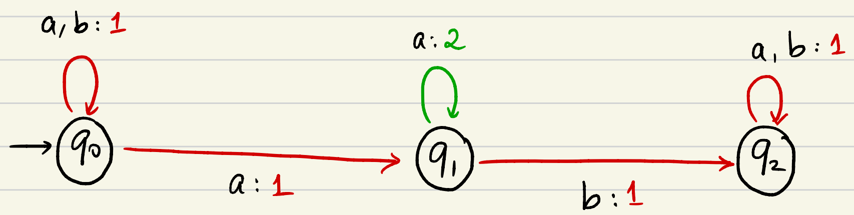
Modification: For any 2-transition of $\mathcal{A}$ which is not in any SCC in $G_{>1}$, we change its priority to 1.

For instance, the transition $q_0 \xrightarrow[a:2]{} q_1$ of $\mathcal{A}_3$ is not in an SCC of $G^{\mathcal{A}_3}_{>1}$.

Therefore we change its priority to 1.

Modified automaton $\mathcal{A}_3'$:

$\mathcal{A}_3'$:



Claim: Modification preserves acceptance of each run.

Let the modified automaton be called $\mathcal{A}'$.

The proof of the claim follows from the following sequence of equivalent statements.

- 1. A run ρ is accepting in $\mathcal{A}$.
- 2. ρ contains finitely many 1-transitions of $\mathcal{A}$.
- 3. ρ eventually stays in an scc of $G_{>1}$.
- 4. ρ sees finitely many 1-transitions in the modified automaton $\mathcal{A}'$.
- 5. ρ is an accepting run in the modified automaton $\mathcal{A}'$.

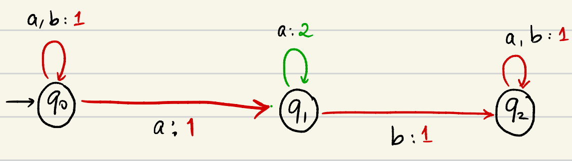
Since this modification does not change acceptance, we make the following assumption.

Assumption: The coBüchi automaton $\mathcal{A}$ is such that all 2-transitions occur in some scc of $\mathcal{A}$.

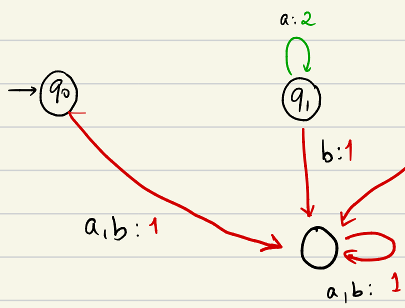
Step 2: The idea of safe-coverage.

$\mathcal{A} \longrightarrow \mathcal{A}_{safe}$

- Add an extra sink state
- all 1-transitions of $\mathcal{A}$ are redirected to the sink state.
- 2-transitions stay as they are.
- On the sink state, there is 1-self loop on every letter.



$\mathcal{A}$



$\mathcal{A}_{safe}$

(only way to accept a word from a state is by remaining in green transitions always)

Safe coverage:

$\mathcal{A}$ has safe coverage if for every state 'q', there is a state 'p' weakly coreachable to q in $\mathcal{A}$ s.t. Eve wins $G1(p;q)$ in $\mathcal{A}_{safe}$.

Step 3:

Recall: we have an automaton $\mathcal{A}$ on which Eve wins 1-token game from everywhere.

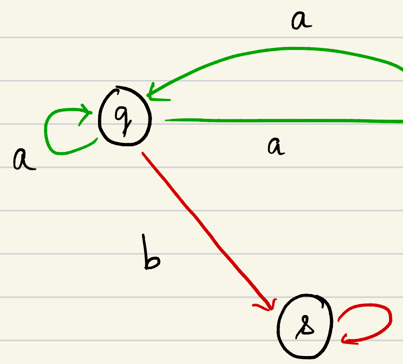
Our aim is to show that $\mathcal{A}$ is HD.

As an intermediate step we will show that $\mathcal{A}$ has safe-coverage.

Recall: We have assumed that all 2-transitions of $\mathcal{A}$ occur in an SCC of $G_{\geq 1}$.

Lemma 1: For every state 'q' in $\mathcal{A}$, there is another state 'p' weakly coreachable to 'q' in $\mathcal{A}$ s.t. Eve wins $G1(p;q)$ in $\mathcal{A}_{safe}$.

Before going into the proof of the lemma, notice that Eve need not have to win $G1(q;q)$ in $\mathcal{A}_{safe}$. Here is an example:



Adam's strategy at 'q': (Eve is also at q)

- 1. Play 'a'.
- 2. if Eve plays $q \xrightarrow{a} q$, Adam moves to r and then plays 'b'.
- 3. if Eve plays $q \xrightarrow{a} r$, Adam stays at 'q' and then plays 'ab'.

Proof of Lemma 1:

- Suppose Adam wins $G1(p;q)$ for all $p \in WCR(A,q)$ in A_{safe}

weakly coreachable states of q in A.

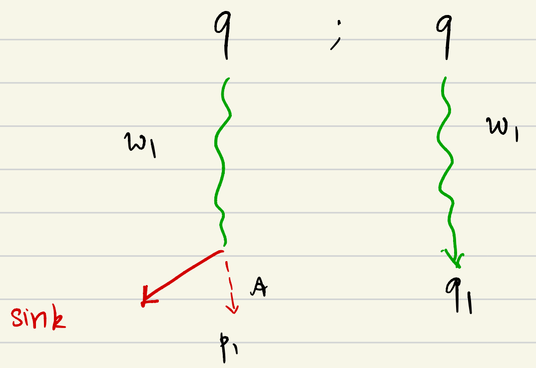
- We will show that Adam wins $G1(q;q)$ in A

↳ a contradiction to hypothesis that Eve wins from everywhere in A.

in particular, Adam wins $G1(q;q)$ in A_{safe}.

Let σ be Adam's winning strategy.

- What happens when Adam plays σ in $G1(q;q)$ in A?



$\exists\, w_1 \in \Sigma^+$ s.t. Eve is forced to take a 1-transition leading to sink in A_{safe}, whereas Adam continues in green.

In A this transition leads to state p_1 (as shown above).

- By our assumption about 2-transitions, all 2-transitions lie inside an SCC.

- Therefore there is a path from q_1 to q on green transitions in A / A_{safe}. Let the word be w_1'

- If Eve is unable to continue on w_1' from p_1, we are done.

- Otherwise:

We Eve reaches a state $s_1 \in WCR(A, q)$ on $w_1 w_1'$.

Continuing this argument, Adam is able to generate a green run.

Whereas, Eve's run contains infinitely many red transitions.

- Hence Adam wins $G1(q; q)$ → a contradiction.

Lemma 2: For each state q in $\mathcal{A}$, there is a state $p \in WCR(\mathcal{A}, q)$ s.t.

Eve wins $G1(p; q)$ and $G1(p; p)$ in $\mathcal{A}_{safe}$.

Proof:

→ Draw a directed graph with states as vertices.

→ Put an edge $q_1 \longrightarrow p_1$ if $p_1 \in WCR(\mathcal{A}, q_1)$ and Eve wins $G1(p_1; q_1)$ in $\mathcal{A}_{safe}$.

→ From q there is a path

$$q \longrightarrow p_1 \longrightarrow p_2 \longrightarrow \cdots\cdots r$$

Then $r \in WCR(\mathcal{A}, q)$ and

Eve wins $G1(r; q)$ and $G1(r; r)$

as required.

This is because from Lemma 1, every vertex has outdegree ≥ 1.

– Rest of the proof in the next lecture.