HISTORY DETERMINISM

Lecture 6: Eve wins 2-token game implies there is a simulation-equivalent subautomaton in which Eve wins from everywhere - Part II

Reference: Section 3.3 of Keya Prakash's PhD thesis

The goal of this lecture is to prove the following theorem.

Theorem: Let $\mathcal{A}$ be a parity automaton on which Eve wins the 2-token game.

Then, there is a simulation-equivalent subautomaton $\mathcal{B}$ of $\mathcal{A}$ s.t. the following three conditions hold:

- 1. For each $k \geq 1$, Eve wins the k-token game from everywhere in $\mathcal{B}$.
- 2. For states $q, p_1, p_2, \ldots, p_k$ that are weakly co-reachable in $\mathcal{B}$, Eve wins the k-token game $(q; p_1, p_2, \ldots, p_k)$ in $\mathcal{B}$.
- 3. If $\mathcal{B}$ is HD, then so is $\mathcal{A}$.

The proof proceeds in the following steps:

Step 1: Construct a specific subautomaton $\mathcal{B}$

Step 2: Show that this subautomaton satisfies a "stronger" claim.

Step 3: Derive items 1 and 3 of the theorem from Step 2.

Step 4: Derive item 2 from step 2.

Step 1: Construct a specific subautomaton $\mathcal{B}$.

We are given that Eve wins 2-token game on $\mathcal{A}$; that is:

$$\boxed{\text{Eve wins } G2\left((\mathcal{A},q_0);\ (\mathcal{A},q_0),(\mathcal{A},q_0)\right)}$$

Consider an arbitrary winning strategy σ of Eve in this game. This strategy σ may 'use' only a subset of transitions of $\mathcal{A}$.

- We say σ uses a transition $p \xrightarrow{a} p'$ if for some finite play of the game that ends with Eve's token at p, strategy σ instructs to take $p \xrightarrow{a} p'$ as the next transition.

- When we restrict $\mathcal{A}$ to only the transitions used by σ, we get a subautomaton $\mathcal{B}_\sigma$.

- Any winning strategy can be chosen to get a subautomaton. However, keeping Step 2 in mind, a particular winning winning strategy is chosen.

Choice of winning strategy:

- Since Eve wins 2-token game on $\mathcal{A}$, Eve also wins the 3-token game on $\mathcal{A}$.

- Let σ_3 be Eve's winning strategy in the 3-token game.

i.e., : σ_3 is a winning strategy for Eve in

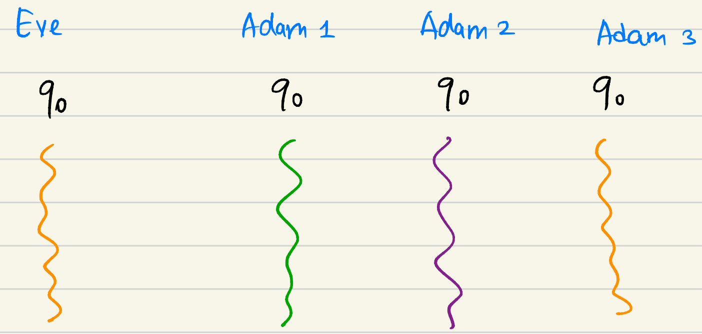
$$G3\left((\mathcal{A}, q_0) ; (\mathcal{A}, q_0) , (\mathcal{A}, q_0) , (\mathcal{A}, q_0) \right)$$

From σ_3 we can derive a winning strategy for the 2-token game :

→ Consider the 2-token game.

→ Assume a dummy 3rd token for Adam in which Eve's token is simply copied.

$\therefore$



At any history, σ_3 gives a move for Eve in this game.

→ Let σ_2 be this strategy (that uses a dummy token + σ_3).

The subautomaton $\mathcal{B}$: We choose $\mathcal{B}$ to be the subautomaton $\mathcal{B}_{\sigma_2}$.

→ all transitions used by σ_2.

Step 2: A stronger claim on this subautomaton.

Claim: For every finite word u and states p, r_1, r_2 s.t.

- there is a run on u from q_0 to p in $\mathcal{B}$

$$\left(q_0 \xrightarrow[\mathcal{B}]{u} p \right)$$

- and there are runs on u from q_0 to r_1 and r_2 in $\mathcal{A}$

$$\left(q_0 \xrightarrow[\mathcal{A}]{u} r_1, \quad q_0 \xrightarrow[\mathcal{A}]{u} r_2 \right)$$

Eve wins $G3\left((\mathcal{B}, p) ; (\mathcal{A}, r_1), (\mathcal{A}, r_2), (\mathcal{A}, p) \right)$

- in this game, Eve should move her token only along $\mathcal{B}$.
- Adam can move along $\mathcal{A}$ in all his tokens.
- His third token initially starts at the same place as Eve's initial token.

Proof of this claim proceeds by an induction on the length of u.

Base case: $u = \varepsilon$. To show: Eve wins

$$G3 \left((\mathcal{B}, q_0) ; (\mathcal{A}, q_0), (\mathcal{A}, q_0), (\mathcal{A}, q_0) \right)$$

- Consider strategy σ_3 discussed in Step 1. We cannot apply it here because σ_3 may use transitions not in $\mathcal{B}$.

- We only know that σ_2 uses transitions along $\mathcal{B}$. So we need to try proving the claim using σ_2.

We know that using σ_2:

$$\boxed{\text{Eve wins } G2 \left((\mathcal{B}, q_0) ; (\mathcal{A}, q_0), (\mathcal{A}, q_0) \right)} \rightarrow (1)$$

We know that using σ_3:

$$\boxed{\text{Eve wins } G3 \left((\mathcal{A}, q_0) ; (\mathcal{A}, q_0), (\mathcal{A}, q_0), (\mathcal{A}, q_0) \right)} \rightarrow (2)$$

From (1), we can derive:

$$\boxed{\text{Eve wins } G1 \left((\mathcal{B}, q_0) ; (\mathcal{A}, q_0) \right)} \longrightarrow (3)$$

From (3) and (2), we can conclude:

$$\boxed{\text{Eve wins } G3 \left((\mathcal{B}, q_0) ; (\mathcal{A}, q_0), (\mathcal{A}, q_0), (\mathcal{A}, q_0) \right)}$$

On Adam's moves, Eve considers the strategy σ_3 (as in (2))

This strategy moves the token along $\mathcal{A}$. Now Eve uses the strategy of (3) to determine the final move along $\mathcal{B}$.

Induction case: Assume that for all words of some length k, the claim is true.

Let $u_{k+1} = u_k \cdot a$ with $|u_k| = k$

Let

$$q_0 \xrightarrow[\mathcal{B}]{u_{k+1}} p$$

$$q_0 \xrightarrow[\mathcal{A}]{u_{k+1}} r_1$$

$$q_0 \xrightarrow[\mathcal{A}]{u_{k+1}} r_2$$

To show: Eve wins $G3\left((\mathcal{B}, p) ; (\mathcal{A}; r_1), (\mathcal{A}, r_2), (\mathcal{A}, p)\right)$

Let $q_0 \xrightarrow[\mathcal{B}]{u_k} pre(p)$

$$q_0 \xrightarrow[\mathcal{A}]{u_k} pre(r_1)$$

$$q_0 \xrightarrow[\mathcal{A}]{u_k} pre(r_2)$$

By induction hypothesis:

$$\boxed{\text{Eve wins } G3\left((\mathcal{B}, pre(p)); (\mathcal{A}, pre(r_1)), (\mathcal{A}, pre(r_2)), (\mathcal{A}, pre(p))\right)}$$

Suppose the next move of Eve at this position in this game is:

$$pre(p) \xrightarrow{a} p'$$

This is a transition in $\mathcal{B}$.

No matter what Adam does to his tokens, Eve can win from p'.

In particular, if Adam moves his tokens to:

$$r_1, \quad r_2, \quad p$$

Eve can still win from p'.

$\therefore$ $\boxed{\text{Eve wins } G3\,((\mathcal{B}, p');\ (\mathcal{A}, r_1), (\mathcal{A}, r_2), (\mathcal{A}, p))}$

$\longrightarrow 1$

- Notice that we have considered two transitions of $\mathcal{B}$ so far:

$$
\begin{tikzcd}
 & pre(p) \arrow[dl, "a"'] \arrow[dr, "a"] & \\
p & & p'
\end{tikzcd}
$$

$\rightarrow$ Since $pre(p) \xrightarrow{a} p$ appears in $\mathcal{B}$, it is used by strategy σ_2.

$\therefore$ there is a position q_1, q_2 for Adam with Eve in $pre(p)$ where σ_2 takes the transition $pre(p) \xrightarrow{a} p$.

Recall how we obtained σ_2. We obtained it from σ_3 assuming Adam's third token is a copy of Eve's token.

$\therefore$ in the configuration:

$$pre(p)\ ;\ q_1, q_2, pre(p)$$

Eve plays $pre(p) \xrightarrow{a} p$ to win.

In particular, if Adam takes the following transition on his third token:

$$pre(p) \xrightarrow{a} p'$$

still Eve wins.

In summary,

$$\boxed{\text{Eve wins } G3\big((\mathcal{A}, p);\ (\mathcal{A}, suc(q_1)), (\mathcal{A}, suc(q_2)), (\mathcal{A}, p')\big)}$$ ②

where $suc(q_1)$, $suc(q_2)$ are some arbitrary a-successors of q_1, q_2.

From ①, we can also conclude:

$$\boxed{\text{Eve wins } G3\big((\mathcal{A}, p');\ (\mathcal{A}, r_1), (\mathcal{A}, r_2), (\mathcal{A}, p)\big)}$$ ③

From ② and ③, we conclude:

Eve wins $G3\big((\mathcal{A}, p);\ (\mathcal{A}, r_1), (\mathcal{A}, r_2), (\mathcal{A}, p)\big)$

which proves the induction step.

Step 3: Deriving items 1 and 3 of the theorem from Step 1.

Step 1 identified the subautomaton $\mathcal{B}$ and

Step 2 showed that for every finite word u, and states

p (reachable in $\mathcal{B}$) and q_1, q_2 (reachable in $\mathcal{A}$) on u,

Eve wins 2-token game

$$G3\,((\mathcal{B}, p) \,;\, (\mathcal{A}, r_1), (\mathcal{A}, r_2), (\mathcal{A}, p))$$

→ Let $\hat{\sigma}$ be Eve's winning strategy in the above game.

To show: Eve wins from everywhere on $\mathcal{B}$.

Pick a word $u \in \Sigma^*$

Suppose there are runs in $\mathcal{B}$ starting from q_0

that lead to p, r_1, r_2.

We know Eve wins $G3((\mathcal{B}, p) \,;\, (\mathcal{A}, r_1)\,(\mathcal{A}, r_2)\,(\mathcal{A}, p))$

$\implies$ Eve wins $G3\,((\mathcal{B}, p) \,;\, (\mathcal{A}, r_1)\,(\mathcal{A}, r_2))$

$\implies$ Eve wins $G3\,((\mathcal{B}, p) \,;\, (\mathcal{B}, r_1)\,(\mathcal{B}, r_2))$

↳ as required.

To show that $\mathcal{B}$ is simulation equivalent:

Since $\mathcal{B}$ is a subautomaton of $\mathcal{A}$, $\mathcal{A}$ simulates $\mathcal{B}$ trivially.

Need to show $\mathcal{B}$ simulates $\mathcal{A}$, ie., $\mathcal{B}$ wins $Sim(\mathcal{B}, \mathcal{A})$.

From Step 2, we can conclude Eve wins $G1((\mathcal{B}, q_0) \,;\, (\mathcal{A}, q_0))$

If Eve wins $G1\,((\mathcal{B}, q_0) \,;\, (\mathcal{A}, q_0))$ then Eve wins $Sim(\mathcal{B}, \mathcal{A})$

by using same strategy of G1 and

ignoring Adam's last transition

played in Sim.

∴ $\mathcal{B}$ simulates $\mathcal{A}$.

To show: If $\mathcal{B}$ is HD, so is $\mathcal{A}$.

In the notes of previous lecture, we have proved that if $\mathcal{A}$ and $\mathcal{B}$ are simulation-equivalent, then

$\mathcal{A}$ is HD $\iff$ $\mathcal{B}$ is HD.

As we have shown that $\mathcal{A}, \mathcal{B}$ are simulation equivalent, item 3 follows.

Note: Item 1 of Theorem 1 follows from these discussions,

- for $k=1$, $k=2$, we have seen that Eve wins from everywhere in $\mathcal{B}$.

By an analogous argument where we showed that Eve wins 2-token $\Rightarrow$ Eve wins k-token, $k \geq 3$ we can conclude that Eve wins k-token game $\forall k \geq 3$ too from everywhere in $\mathcal{B}$.

Part 4: Weak co-reachability.

→ Consider a parity automaton $\mathcal{A}$.

→ We say states p, q are co-reachable in $\mathcal{A}$ if

$\exists u \in \Sigma^*$ and runs $q_0 \xrightarrow[\mathcal{A}]{u} p$

$q_0 \xrightarrow[\mathcal{A}]{u} q$

→ We write $(p,q) \in CR$ in that case.

* WCR is the reflexive & transitive closure of the CR relation.

p, q are weakly co-reachable if $(p,q) \in WCR$.

∴ If $(p,q) \in WCR$, there are 'intermediate' states $q_1, q_2, \ldots, q_m$ s.t.

$(p, q_1) \in CR$, $(q_1, q_2) \in CR, \ldots$ $(q_m, q) \in CR$.

→ We say a set of states $q, p_1, p_2, \ldots, p_k$ is weakly co reachable

if every pair of them is weakly co-reachable.

Lemma: Let $\mathcal{A}$ be a parity automaton, and p, q, r be states s.t.

$(p,q) \in CR$. and $(q,r) \in CR$.

Then Eve wins $G1(p;q)$ and $G1(q,r)$

implies Eve wins $G1(p;r)$.

Part a: To show: If $q, p_1, p_2, \ldots, p_k$ are weakly coreachable in $\mathcal{B}$, then Eve wins

$$G_k\,((\mathcal{B}, q) ; (\mathcal{B}, p_1)\,(\mathcal{B}, p_2) \ldots, (\mathcal{B}, p_k))$$

Firstly, since Eve wins k-token game from everywhere in $\mathcal{B}$, we can conclude:

$$\boxed{\text{Eve wins } G_k\,((\mathcal{B},q) ; (\mathcal{B},q)\,(\mathcal{B},q) \ldots (\mathcal{B},q))} \longrightarrow (1)$$

Lemma: Eve wins $G_1\,((\mathcal{B}, q) ; (\mathcal{B}, p_i))\quad \forall i$

Proof: As q, p_i are WCR, there are intermediate states

$$q = r_0, r_1, r_2, \ldots, r_m = p_i \quad \text{s.t.} \quad (r_i, r_{i+1}) \in CR.$$

As Eve wins G_1 from everywhere in $\mathcal{B}$, Eve wins $G_1(r_i, r_{i+1})$

By transitivity, Eve wins $G_1(q, p_i)$.

From (1) and the above lemma, we can conclude that

Eve wins $G_k(q ; p_1, \ldots, p_k)$.