HISTORY DETERMINISM

Lecture 5: Eve wins 2-token game implies there is a simulation-equivalent subautomaton in which Eve wins from everywhere — Part I

Reference: Section 3.3 of Keya Prakash's PhD thesis

The goal of this lecture is to understand the statement of the following theorem.

Theorem: Let $\mathcal{A}$ be a parity automaton on which Eve wins the 2-token game.

Then, there is a simulation-equivalent subautomaton $\mathcal{B}$ of $\mathcal{A}$ s.t. the following three conditions hold:

- 1. For each $k \geq 1$, Eve wins the k-token game from everywhere in $\mathcal{B}$.
- 2. For states $q, p_1, p_2, \dots, p_k$ that are weakly co-reachable in $\mathcal{B}$, Eve wins the k-token game $(q; p_1, p_2, \dots, p_k)$ in $\mathcal{B}$.
- 3. If $\mathcal{B}$ is HD, then so is $\mathcal{A}$.

Part 1: The simulation game, relation to 'guidability'

Part 2: Token games on different automata: notations and some properties

Part 3: Weak coreachability

Part 4: Illustration of the above theorem on an example

Part 1: Simulations & the simulation game

Consider two parity automata $\mathcal{A}$ and $\mathcal{B}$

$$\mathcal{A} = (P, \Sigma, p_0, \Delta_A)$$

$$\mathcal{B} = (Q, \Sigma, q_0, \Delta_B)$$

$\mathcal{B}$ simulates $\mathcal{A}$: We say that

$$\boxed{\mathcal{B} \text{ simulates } \mathcal{A} \text{ if Eve wins the simulation game } Sim(\mathcal{B}, \mathcal{A})}$$

The simulation game $Sim(\mathcal{B}, \mathcal{A})$: The simulation game of $\mathcal{A}$ by $\mathcal{B}$, denoted as $Sim(\mathcal{B}, \mathcal{A})$ is a 2-player game played between Eve and Adam as follows. Eve and Adam place a token initially at q_0 and p_0 respectively. The game proceeds for infinitely many rounds, and in round i where Eve's token is at q_i and Adam's token is at p_i:

- 1. Adam chooses a letter a_i.
- 2. Adam chooses a transition $p_i \xrightarrow{a_i} p_{i+1}$ in $\mathcal{A}$.
- 3. Eve chooses a transition $q_i \xrightarrow{a_i} q_{i+1}$ in $\mathcal{B}$.

Eve wins a play if the following holds:

if Adam's run is accepting, then Eve's run is accepting as well.

If Eve has a strategy to win $Sim(\mathcal{B}, \mathcal{A})$, then we say Eve wins $Sim(\mathcal{B}, \mathcal{A})$, and hence $\mathcal{B}$ simulates $\mathcal{A}$.

Lemma: If $\mathcal{B}$ simulates $\mathcal{A}$, then $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$.

Question: Is $Sim(\mathcal{B}, \mathcal{A})$ determined?

Yes. This can be seen as a 2-dimensional parity game. Hence we get the following stronger property.

Proposition: For every two parity automata $\mathcal{A}$ and $\mathcal{B}$, either Eve or Adam wins $Sim(\mathcal{B}, \mathcal{A})$, and the winner has a finite-memory winning strategy. Additionally, if Eve wins $Sim(\mathcal{B}, \mathcal{A})$, then she has a positional winning strategy.

Lemma: For every three parity automata $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, if $\mathcal{B}$ simulates $\mathcal{A}$ and $\mathcal{C}$ simulates $\mathcal{B}$, then $\mathcal{C}$ simulates $\mathcal{A}$.

Proof sketch: In $Sim(\mathcal{C}, \mathcal{A})$, Eve can keep in its memory, a token that takes transitions in $\mathcal{B}$. It can use strategy σ_{BA} of $Sim(\mathcal{B}, \mathcal{A})$ to update the memory token, and σ_{CB} of $Sim(\mathcal{C}, \mathcal{B})$ to update the actual token.

Next we will relate simulations to a notion called 'guidability'.

Guidable Automata

Definition: A non-deterministic parity automaton $\mathcal{A}$ is said to be guidable if for all automata $\mathcal{B}$ with $\mathcal{L}(\mathcal{B}) \subseteq \mathcal{L}(\mathcal{A})$, $\mathcal{A}$ simulates $\mathcal{B}$.

We want to show the following theorem.

Theorem: For every parity automaton $\mathcal{A}$,

$$\boxed{\mathcal{A} \text{ is HD} \iff \mathcal{A} \text{ is guidable}}$$

HD implies guidable:

Suppose $\mathcal{A}$ is HD. Then Eve has a strategy σ_{HD} to win the letter game $HD(\mathcal{A})$.

Consider some automaton $\mathcal{B}$ s.t. $\mathcal{L}(\mathcal{B}) \subseteq \mathcal{L}(\mathcal{A})$.

We need to show that Eve wins $Sim(\mathcal{A}, \mathcal{B})$.

→ Eve can play σ_{HD} in $Sim(\mathcal{A}, \mathcal{B})$ without worrying about Adam's token.

→ For a word w, if Adam's token is accepting, then $w \in \mathcal{L}(\mathcal{B})$. Since $\mathcal{L}(\mathcal{B}) \subseteq \mathcal{L}(\mathcal{A})$, we also have $w \in \mathcal{L}(\mathcal{A})$. Therefore σ_{HD} would give an accepting run.

This shows σ_{HD} is a winning strategy for Eve in $Sim(\mathcal{A}, \mathcal{B})$, and hence $\mathcal{A}$ simulates $\mathcal{B}$.

$\Rightarrow$ $\mathcal{A}$ is guidable.

Guidable implies HD:

Suppose $\mathcal{A}$ is guidable.

Then for all $\mathcal{B}$ s.t. $\mathcal{L}(\mathcal{B}) \subseteq \mathcal{L}(\mathcal{A})$, $\mathcal{A}$ simulates $\mathcal{B}$.

Let $\mathcal{D}$ be a deterministic parity automaton s.t. $\mathcal{L}(\mathcal{D}) = \mathcal{L}(\mathcal{A})$.

→ As $\mathcal{A}$ is guidable, $\mathcal{A}$ simulates $\mathcal{D}$.

$\therefore$ Eve wins $Sim(\mathcal{A}, \mathcal{D})$.

Let σ be Eve's strategy in $Sim(\mathcal{A}, \mathcal{D})$.

→ We claim that σ is a winning strategy in $HD(\mathcal{A})$.

- → On any word, $\mathcal{D}$ has a unique run (assume $\mathcal{D}$ is complete).
- → So in the letter game, for any finite prefix, Eve keeps the corresponding play of $Sim(\mathcal{A}, \mathcal{D})$ and chooses the next transition based on the strategy σ.

This shows $\mathcal{A}$ is HD.

Corollary: If $\mathcal{A}$ is a non-deterministic parity automaton that simulates a language-equivalent HD automaton $\mathcal{H}$, then $\mathcal{A}$ is HD.

Proof:
- $\mathcal{A}$ simulates $\mathcal{H}$.
- $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{H})$
- $\mathcal{H}$ is HD

$\Rightarrow$ $\mathcal{H}$ is guidable

We show $\mathcal{A}$ is guidable:

Pick some $\mathcal{B}$ s.t. $\mathcal{L}(\mathcal{B}) \subseteq \mathcal{L}(\mathcal{A})$.

As $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{A})$, we have:

$$\mathcal{L}(\mathcal{B}) \subseteq \mathcal{L}(\mathcal{H}).$$

As $\mathcal{H}$ is guidable, $\mathcal{H}$ simulates $\mathcal{B}$.

Since $\mathcal{A}$ simulates $\mathcal{H}$, and $\mathcal{H}$ simulates $\mathcal{B}$,

$\mathcal{A}$ simulates $\mathcal{B}$.

$\Rightarrow$ $\mathcal{A}$ is guidable.

Corollary: Let $\mathcal{A}, \mathcal{B}$ be parity automata which are simulation-equivalent.

Then $\mathcal{A}$ is HD $\Leftrightarrow$ $\mathcal{B}$ is HD.

Proof:

Since $\mathcal{A}, \mathcal{B}$ are simulation equivalent, $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{B})$.

$\mathcal{A}$ is HD $\Rightarrow$ $\mathcal{A}$ is guidable

To show $\mathcal{B}$ is guidable.

- Let $\mathcal{C}$ s.t. $\mathcal{L}(\mathcal{C}) \subseteq \mathcal{L}(\mathcal{B})$
- As $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{B})$, we have $\mathcal{L}(\mathcal{C}) \subseteq \mathcal{L}(\mathcal{A})$
- Therefore $\mathcal{A}$ simulates $\mathcal{C}$
- Since $\mathcal{B}$ simulates $\mathcal{A}$, and $\mathcal{A}$ simulates $\mathcal{C}$

  we get $\mathcal{B}$ simulates $\mathcal{C}$

$\Rightarrow$ $\mathcal{B}$ is guidable

$\Rightarrow$ $\mathcal{B}$ is HD.

Hence we have shown: $\mathcal{A}$ is HD $\Rightarrow$ $\mathcal{B}$ is HD.

Other direction is symmetric.

Part 3: Weak co-reachability.

→ Consider a parity automaton $\mathcal{A}$.

→ We say states p, q are co-reachable in $\mathcal{A}$ if

$\exists u \in \Sigma^*$ and runs
$$q_0 \overset{u}{\underset{\mathcal{A}}{\rightsquigarrow}} p$$
$$q_0 \overset{u}{\underset{\mathcal{A}}{\rightsquigarrow}} q$$

→ We write $(p,q) \in CR$ in that case.

* WCR is the reflexive & transitive closure of the CR relation.

p, q are weakly co-reachable if $(p,q) \in WCR$.

$\therefore$ If $(p,q) \in WCR$, there are 'intermediate' states $q_1, q_2, \ldots, q_m$ s.t.
$$(p, q_1) \in CR, \quad (q_1, q_2) \in CR, \ldots \quad (q_m, q) \in CR.$$

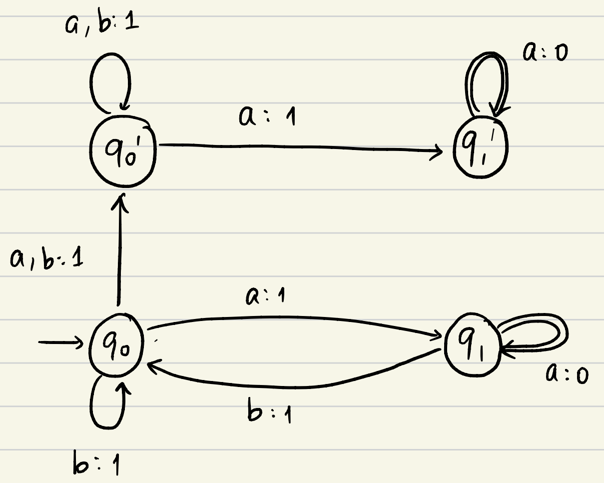
→ We say a set of states $q, p_1, p_2, \ldots, p_k$ is weakly co-reachable if every pair of them is weakly co-reachable.

Lemma: Let $\mathcal{A}$ be a parity automaton, and p, q, r be states s.t. $(p,q) \in CR$ and $(q, r) \in CR$.

Then Eve wins $G1(p; q)$ and $G1(q, r)$ implies Eve wins $G1(p; r)$.

Part 4: An example.

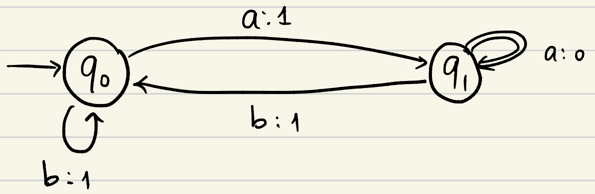
Consider this automaton $\mathcal{A}$:



$\leftarrow$ is HP

Eve wins 2-token game on $\mathcal{A}$. However it does not win from everywhere.

Consider this subautomaton:



Eve wins 2-token from everywhere.

The only reachable configurations are:

(q_0, q_0, q_0) and (q_1, q_1, q_1)

- Here it turned out to be deterministic, but that's not necessarily the case in general.
- For instance, what happens if there is an additional self-loop $q_0 \xrightarrow{a} q_0$?